Some Fixed Point Theorems for Generalized Modular Metric Spaces

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Abstract: *In this paper, we obtain some fixed point theorems in generalized modular metric spaces and obtain some examples in support of our results.*

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Classification:54H25, 47H10.

1. Introduction

Fixed point theory in Modular function spaces was initiated by Khamsi, Kozlowski, and Reich [1]. Modular function spaces area was introduced by Nakano [7] which is a special case of the theory of modular vector spaces. Modular metric spaces were introduced in [3,4]. Fixed point theory in modular metric spaces was studied by Abdou andKhamsi [2]. In this paper, we follow the same approach as the one used in generalizations of standard metric spaces allows for some deep understanding of the classical results obtained in metric spaces. One has to be always careful when coming up with a new generalization. This is the case with the generalized metric distance introduced by Jleli and Samet in [6]. They showed that this generalization encompasses metric spaces, b-metric spaces, dislocated metric spaces, and modular vector spaces[9,10,11]. In this paper, considering both a modular metric space and a generalized metric space in the sense of Jleli and Samet [6], we used the concept of generalized modular metric space. Then we are using the Banach contraction principle (BCP) and Ćirić's fixed point theorem for quasi-contraction mappings in this new space. We take the contraction constant k < 1,c ,
First, we give the definition of generalized modular metric spaces.

2 Preliminaries

Definition 2.1. Let X be an abstract set. A function $D: X \times X \to [0,\infty]$ is said to be a generalized modular metric (GMM) on X if it satisfies the following three axioms:

GMM1: If $D(x,y) = 0$ then $x = y$

GMM2: $D(x,y) = D(y,x)$; for$x, y \in X$

GMM3: There exists $c \geq 1$ such that if $(x,y) \in X, \{x_n\} \subset X$ with $\lim_{n\to\infty} D(x_n, x) = 0$ then either $\lim_{n\to\infty} Sup\, D(x_n, y) = \infty$ or $D(x,y) \leq c \lim_{n\to\infty} Sup\, D(x_n, y) < \infty$.

The pair $(X, D)$is said to be generalized modular metric space (GMMS).

It is easy to check that if there exist $x, y \in X$ such that $\{x_n\} \subset X$ with $\lim_{n\to\infty} D(x_n, x) = 0$ and $D(x,y) < \infty$, then we must have $c \geq 1$

Example: Let $X = \{0,1,2, \ldots..\}$ and $D(0,n) = 1/n$ if $n \geq 1$ $D(0,0) = 0$ and $D(m,n) = \infty$ if $m, n > 0$. Then (X, D) is a generalized modular metric space.

Definition 2.2. Let (X, D) be a GMMS

1. A sequence $\{x_n\}_{n\in N}$ in X is said to be D-convergent to $x \in X$ if $D(x_n, x) \to 0$ as $n \to \infty$

2. The sequence $\{x_n\}_{n\in N}$ in X is said to be $D-$ Cauchy if $D(x_m, x_n) \to 0$ as $m, n \to \infty$
3. A subset C of X is said to be D-Closed if for any $\{x_n\}$ from C which D-Converges to x, $x \in C$
4. A subset C of $X$is said to be D – Complete if for D – Cauchy sequence $\{x_n\}$ in C $\lim_{n,m\to\infty} D(x_n, x_m) = 0$ there exists a point x in C such that $\lim_{n,m\to\infty} D(x_n, x) = 0$
5. A subset C of X is said to be D – bounded if we have $\delta(C) = Sup\,\{D(x,y): x, y \in C\} < \infty$.

3. Main Results

Proposition 3.1. Let (X, D) be a GMMS. Let $\{x_n\}$ be a sequence in X. Let $(x, y) \in X \times X$ such that $D(x_n, x) \to 0$ and $D(x_n, y) \to 0$ as $n \to \infty$. Then $x = y$.

Proof: Using the property GMM3, we have $D(x, y) \le c \lim_{n\to\infty} Sup\, D_n(x_n, y) = 0$,

which implies from the property GMM1 that $x = y$.

The Banach Contraction Principle (BCP) in GMMS:

Definition 3.2. Let (X, D) be a GMMS and $f: X \to X$ be a mapping f is called a D – Contraction mapping if there exists $k \in (0,1)$ such that $D\big(f(x), f(y)\big) \le kD(x, y)$ for any $(x, y) \in X \times X$. Then x is said to be a fixed point if $f(x) = x$.

Proposition 3.3. Let (X, D) be a GMMS. Let $f: X \to X$be a D – contraction mapping. If w_1 and $w_2$are fixed points of f and $D(w_1, w_2) < \infty$ then we have $w_1 = w_2$.

Proof: Let $w_1, w_2 \in X$ be two fixed points of f such that D$(w_1, w_2) < \infty$.

As f is a D – contraction, there exists $k \in (0,1)$

such that $D(w_1, w_2) = D\big(f(w_1), f(w_2)\big) \le kD(w_1, w_2)$

since $D(w_1, w_2) < \infty$

We conclude that $D(w_1, w_2) = 0$

Which implies that $w_1 = w_2$ from GMM1

Definition 3.4. Let (X, D) be a GMMS and $f: X \to X$ be a mapping. For any $x \in X$ define the Orbit of x by O(x) = { $x, f(x), f^2(x), \ldots\ldots$}

Write $\delta(x) = Sup\,\{D\big(f^n(x), f^t(x)\big): n, t \in N\}$

Theorem 3.5. Let (X, D)be a GMMS. Assume that X is D – Complete. Let $f: X \to X$ be a D – Contraction mapping. Assume that $\delta(x_0)$ is finite for some $x_0 \in X$. Then $\{f^n(x_0)\}D$ – converges to a fixed point *w* of *f*. Moreover, if *D(x,w)*<∞ for some $x \in X$ then $\{f^n(x)\}D$ – converges to *w*.

Proof: Let $x_0 \in X$ be such that $\delta(x_0) < \infty$.

Then $D(f^{n+p}(x), f^n(x_0) \le k^n D(f^p(x_0), x_0) \le k^n \delta(x_0)$ for any *n*, $p \in N$.

Since *k*< 1, $\{f^n(x_0)\}$ is *D* – Cauchy.

As X is D – complete, then there exists *w* in D such that $\lim_{n\to\infty} D(f^n(x_0), w) = 0$

Since
$D(f^n(x_0), f(w) \le kD(f^{n-1}(x_0), w)$; 1,2,3, ….

We have $\lim_{n\to\infty} D(f^n(x_0), f(w)) = 0$.

By proposition 1 implies that *f(w)=w*

That is *w* is a fixed point of *f*.

Let $x \in X$ be such that *D*(*x*,*w*) < ∞.

Then $D(f^n(x), w) = D(f^n(x), f^n(w)) \le k^n D(x, w)$ for any *n* ≥ 1.

Since *k*< 1, we get $\lim_{n\to\infty} D(f^n(x),\ w) = 0$

i.e $\{f^n(x)\}$ D – Converges to *w*.

If D(*x*,*y*) < ∞ for all *x*, *y* ∈ *X*,

Then f hasat most one fixed point.

Moreover, if X is D – complete and $\delta(x) < \infty$ for any *x* ∈ *X*,

then all orbits D – converge to the unique fixed point of *f*.

In metric space, *d(x, y)* is always finite.

Because of this reason any contraction will have atmost one fixed point.

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Moreover the orbits of the contraction are all bounded.

Indeed, let $f: M \to M$ be a contraction,

where M is a metric space endowed with a metric distance d,

we have $d(f^{n+1}(x), f^n(x)) < k^n(f(x), x)$; for any $n \in N$ and $x \in M$,

which implies by using triangle inequality

$$d\left(f^{n+p}(x), f^n(x)\right) \leq \sum_{k=0}^{p-1} d\left(f^{n+k+1}(x), f^{n+k}(x)\right)$$

$$\leq \sum_{k=0}^{p-1} k^{n+1} d\left(f \quad (x), x\right)$$

$$\leq \frac{1}{1-k} d(f(x), x); \text{ Since } k<1$$

Hence $Sup\{d(f^n(x), f^n(t); n, t \in N\} \leq \frac{1}{1-k} d(f(x), x) < \infty$ for any $x \in M$.

Ciric Quasi contraction in generalized modular metric spaces

Definition 3.6. Let*(X, D)* be a GMMS. The mapping $f: X \to X$ is said to be a D – quasi contraction if there exists $k \in (0,1)$ such that

$$D\left(f(x), f(y)\right) \leq k\, max\{D(x,y), D\left(x, f(x)\right), D\left(y, f(y)\right), D\left(x, f(y)\right), D\left(y, f(y)\right)\}$$

For any $(x, y) \in X \times X$

Proposition 3.7. Let (X, D) be a GMMS. Let $f: X \to X$ be a D – quasi contraction mapping. If *w* is a fixed point of *f* such that $D(w, w) < \infty$, then we have $D(w, w) = 0$. Moreover, if w_1, w_2 are two fixed points of *f*, *such* that $D(w_1, w_2) < \infty, D(w_1, w_1) < \infty$ *and* $D(w_2, w_2) < \infty$ then we have $w_1 = w_2$.

Proof: Let *w* be a fixed point of *f*, then

D(w,w)=D(f(w),f(w))

$$\leq k \max\{D(w,w), D\left(w, f(w)\right), D\left(w, f(w)\right), D\left(w, f(w)\right), D\left(w, f(w)\right)\}$$

=kD(w,w)

Since *k < 1* and *D(w,w)<∞* , we have *D(w,w)=0*

Let $w_1, w_2 \in X$ be two fixed points of f

such that $D(w_1, w_2) < \infty, D(w_1, w_1) < \infty$ and $D(w_2, w_2) < \infty$.

Since f is a quasi contraction, there exists *k < 1* such that

$$D(w_1, w_2) = D\left(f(w_1), f(w_2)\right)$$

$$\leq k \max\{D(w_1, w_2), D\left(w_1, f(w_1)\right), D(w_2, f(w_2)), D(w_1, f(w_2)), D(w_2$$

$$= k \max\{D(w_1, w_2), D(w_1, w_1), D(w_2, w_2)\}$$

Since $D(w_1, w_1) < \infty, D(w_2, w_2) < \infty$ then $D(w_1, w_1) = D(w_2, w_2) = 0$.

Now we have $D(w_1, w_2) \leq kD(w_1, w_2)$ since $D(w_1, w_2) < \infty$ *and* $k < 1$, we have

$D(w_1, w_2) = 0$. Therefore $w_1 = w_2$

Theorem 3.8. Let (X, D) be a generalized modular metric space. Let $f: X \to X$ be a *D*- quasi contraction mapping. Assume that $k < \frac{1}{c}$, where *c* is constant from GMM3 and there exists $x_0 \in X$ such that $\delta(x_0) < \infty$. Then $\{f^n(x_0)\}$ *D*- converges to some $w \in X$. If $\lim_{n\to\infty} \sup (f^n(x_0), f(w)) < \infty$, then *w* is a fixed point of *f*

Proof: Let f be a D- quasi contraction.

Then $\exists k \in (0,1), \ni, \forall\, p, x, n \in N$ *and* $x \in X$,

we have

$$D\left(f^{n+p+1}(x), f^{n+r+1}(x)\right) \leq kmax\left\{D\left(f^{n+p}(x), f^{n+r}(x)\right), D\left(f^{n+p}(x), f^{n+p+1}(x)\right),\right.$$

$$\left. D(f^{n+r}(x), f^{n+r+1}(x)), D(f^{n+p}(x), f^{n+r+1}(x)), D(f^{n+r}(x), f^{n+p+1}(x))\right\}$$

Hence $\delta\left(f(x)\right) \leq k\delta(x), for\ any\ x \in X.$

Consequently we have $\delta\left(f^n(x_0)\right) \leq k^n \delta(x_0) \ldots \ldots \ldots (1)$

For any $n \geq 1$ using the above inequality, we get

$D\left(f^n(x_0), f^{n+1}(x_0)\right) \leq \delta\left(f^n(x_0)\right) \leq k^n \delta(x_0) - - - - -$ (2) for every $n, t \in N$

Since $\delta(x_0) < \infty$ and $k < \frac{1}{c} \leq 1$

we have $\lim_{n\to\infty} D(f^n(x_0), f^{n+t}(x_0)) = 0$

Which implies that $\{f^n(x_0)\}$ is a D – Cauchy sequence.

Since X is D- complete, there exists $w \in X$

such that $\lim_{n\to\infty} D(f^n(x_0), w) = 0$

i.e$\{f^n(x_0)\}$ D- converges to w.

Now we assume that $\lim_{n\to\infty} Sup\, D(f^n(x_0), f(w)) < \infty$.

Then *D(w, f(w))* <∞, by GMM3

Suppose *D(w, f(w))*> 0

Now

$D(f^{n+1}(x_0), f(w) \leq$
$k \max\{D(f^n(x_0), w), D(f^n(x_0), f^{n+1}(x_0)), D(w, f(w)), D(f^n(x_0), f(w)), D(w, f^{n+1}(x_0))$

$\leq k \max\{D(w, f(w)), D(f^n(x_0), f(w))\}$ – – – – – –(3) for large n

Hence

$$\lim_{n\to\infty} Sup\, D(f^{n+1}(x_0), f(w)) \leq \max\{kD(w, f(w)), k\ lim\, Sup\, D(f^n(x_0), f(w))\}$$

Write $\alpha = lim\, Sup\, D(f^n(x_0), f(w))$

Then $\alpha < \infty$, by hypothesis.

Then $\alpha \leq \max\{kc\alpha, k\alpha\}$.

Therefore $\alpha = 0$

Therefore $f^n(x_0) \to f(w)$

Now
$D(w, f(w)) \leq c\ lim\, Sup\, D(f^n(x_0), f(w)) = 0$

Therefore *D(w,f(w))* = 0

Hence *w* = *f(w)*, by GMM1.

The following example illustrates our Theorem.

Example: $X = \{0,1,2, \dots \dots .\}, D(0,1) = \frac{1}{n}\ if\ n \geq 1, D(0,0) = 0$ and

$D(m,n) = \infty\ if\ m,n > 0$. Then *(X, D)* is a complete generalized modular metric space and the mapping $f: X \to X$ defined by $f(x) = \begin{cases} x+1 & if\ x \neq 0 \\ x & if\ x = 0 \end{cases}$ is a *D*- quasi contraction on *X*. 0 is the unique fixed point of *f*.

Observation: If $x_n \to x \Longrightarrow D(x_n, x) \longrightarrow 0$

Now there exist c, such that $D(x,y) \leq c \lim_{n\to\infty} D(x_n, x)$

Now take $y = x\ \forall n$, then $D(x,x) \leq c \lim_{n\to\infty} D(x_n, x) = c.0 = 0$.

Therefore $D(x,x) = 0$

Now take $z_n = x\ \forall n$, so that $D(z_n, y) \to 0$

Therefore D(x, x) = 0

Now take $y = x\ \forall n$, so that $D(\mathfrak{z}_n, y) \to 0$.

Then $D(x,y) \leq c \lim_{n\to\infty} Sup\, D(\mathfrak{z}_n, y)) = c \lim_{n\to\infty} Sup\, D(x, y) = c\, D(x,y)$

Since D(x, y) > ∞, we have $c \geq 1$.

Conflict of interest

The authors declare that there is no conflict of interest.

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