ISSN 2321-2926

Periodic Solutions for an equation governing dynamics of a renewable resource subjected to weak Allee effect

G. KIRAN KUMAR

Dept. of Mathematics, Dr. B.R. Ambedkar University, Srikakulam, Etcherla, AP, India

Abstract: ***Abstract*** *In this article the minimum number of positive periodic solutions admitted by a non-autonomous scalar differential equation is estimated. This result is employed to find the minimum number of positive periodic solutions admitted by a model representing dynamics of a renewable resource that is subjected to weak Allee effects in a seasonally varying environment. Leggett-Williams multiple fixed point theorem is used to establish the existence of at least two positive periodic solutions for the considered dynamic equation. Two methods are obtained to establish the existence of periodic solutions. Key results are illustrated through numerical simulation.*

1. Introduction

The term Allee effect seems to have originated from the works of Allee [3, 4]. Allee effect refers to reduction in individual fitness at low population density that can lead to extinction [6, 11, 12]. It is strongly related to the extinction vulnerability of populations. According to [28], any ecological mechanism that can lead to a positive relationship between a component of individual fitness and either the number or density of conspecifics can be termed a mechanism of the Allee effect [19, 31] or depensation [10, 15, 21] or negative competition effect [37].

Most ecosystems experience fluctuations in environmental factors which affect the birth rates, mortality rates, carrying capacities and other vital factors of the species in the ecosystems. In spite of the influence of such environmental fluctuations on species dynamics, the amount of analysis which has been carried out on autonomous growth models is much more than that which has been done on models in which the parameters are allowed to vary with time. In recent years an increasing amount of attention has been paid in both the biological and mathematical literature to the effects that such variations have on growth of species. It is evident that many of the variations

with which the species must cope are regular and periodic. The quality and quantity of food and other vital resources, the occurrence of predation and competition, and the susceptibility or exposure to diseases or other hazards are but a few other examples of things which can affect growth and dynamics of species which can vary regularly [13, 27].

2. The Model

Let us consider the following differential equation representing the dynamics of a population subjected to additive Allee effects.

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k} - \frac{\nu}{1+\omega x}\right) \tag{1}$$

where the positive constants r and k stand for intrinsic growth rate and carrying capacity of the resource, v and ω are constants that indicate the severity of Allee effect that has been modelled [1, 2]. Considering these parameters to be positive constants (which is a realistic assumption), it can be easily observed that (1) always admits the trivial solution as one of the equilibrium solutions and it admits at most two positive equilibrium solutions depending on the values of the remaining parameters. Equilibrium analysis and qualitative behaviour of solutions of (1) is presented in [34]. Assuming that the parameters r,k,v,ω to be constants imply that the growth rate, carrying capacity, the extra mortality due to other factors remain the same throughout and are independent of the seasons. We introduce periodic variations in the growth dynamics by assuming that these parameters are periodic of same period [8, 13, 30].

In this article we are interested in studying the existence of positive periodic solutions of (1) under the assumption that the associated coefficients *r,k,v* and ω are positive periodic functions of same period, *P*. Thus we consider the equation

(2)
$$\frac{dx}{d\tau} = r(\tau)x\left(1 - \frac{x}{k(\tau)} - \frac{\nu(\tau)}{1+\omega(\tau)x}\right).$$

The following lemma transforms the *P* periodic differential equation (2) into another equivalent simpler (involving only two periodic coefficients instead of three) *T* periodic differential equation where *T* could be different from *P*.

Lemma 1. *The transformation* $t = G(\tau) = \int_0^\tau r(s)ds$ *transforms (2) to a T - periodic equation given by*

(3)
$$\frac{dy}{dt} = y\left(1 - \frac{y}{\kappa(t)} - \frac{\eta(t)}{1+m(t)y}\right)$$

with $y(t) = x(G^{-1}(t))$ *where* $\kappa(t) = k(G^{-1}(t))$, $\eta(t) = v(G^{-1}(t))$ and $m(t) = \omega(G^{-1}(t))$ *are positive periodic functions of period* $T = G(P)$. *Also, for each T - periodic solution* $y(t)$ *of (3),* $x(\tau) = x(G^{-1}(t))$ *defines a P - periodic solution of (2).*

In the light of lemma 1, we shall concentrate only on the existence of *T* periodic solutions for (3).

3. Periodic solutions for a general scalar differential equation

In this section we shall study the existence of positive periodic solutions for a general scalar differential equation for which (3) becomes a special case. The results developed in this section will be applied to (3) to find conditions under which existence of two positive periodic solutions are guaranteed.

Let us consider the following general scalar differential equation

(4) $\frac{dy}{dt} = y + f(t,y)$

where *f*, defined on $R \times R$, is a non positive valued continuous function and satisfies $f(t+T,y) = f(t,y)$. We have the following lemma which is easily verifiable.

Lemma 2. *If* $y(t)$ *is a T - periodic solution of (4) then it also satisfies the integral equation*

(5) $y(t) = \int_t^{t+T} G(t,s)f(s,y(s))ds$

where $G(t,s)$ *is the Green's function given by* $G(t,s) = \frac{e^{-(s-t)}}{e^{-T}-1},\ s \in [t,t+T]$.

Note that the Green's function $G(t,s)$ satisfies

(6)
$$\frac{1}{e^{-T}-1} < G(t,s) < \frac{e^{-T}}{e^{-T}-1} < 0,\ s \in [t,t+T].$$

The following fundamentals are needed to prove the results to follow. Let *X* be a Banach space and *K* be a cone in *X*. A mapping $\psi : K \to [0,\infty)$ is said to be concave nonnegative continuous functional [20] on *K* if it is continuous, nonnegative and satisfies

$$\psi(\eta x + (1-\eta)y) \geq \eta\psi(x) + (1-\eta)\psi(y), x,y \in K, \eta \in [0,1].$$

Let $a,b,c > 0$ be constants with *K* and *X* as defined above. Define

$K_a = \{y \in K : \|y\| < a\}$

and

$K(\psi,b,c) = \{y \in K : \|y\| < c,\ \psi(y) > b\}$.

Corollary 1. *Suppose that there exist two positive constants* $c_1 < c_2$ *such that*

$(\tilde{H}_1)$ $\frac{e^{-T}}{e^{-T}-1}\int_0^T f(s,y)ds = y$ at $y = c_2$ *and*

$$\frac{e^{-T}}{e^{-T}-1}\int_0^T f(s,y)ds > c_2 \quad \text{for} \quad c_2 < y \leq \frac{c_2}{e^{-T}}$$

$(\tilde{H}_2)$ $\max_{0<y\leq c_1}\left\{\frac{1}{e^{-T}-1}\int_0^T f(s,y)ds\right\} < c_1$.

Then (4) admits at least two positive T-periodic solutions.

Theorem 3. *Assume that*

T

(H_1^*) $\frac{e^{-T}}{e^{-T}-1}\int_0^T f(s,y)ds$ *is strictly monotonically increasing such that*

$$\lim_{y\to\infty}\frac{e^{-T}}{e^{-T}-1}\frac{1}{y}\int_0^T f(s,y)ds=\infty$$

(H_2^*) *there exists a constant* $c>0$ *such that* $\frac{1}{e^{-T}-1}\int_0^T f(s,y)ds<c$ *in a small left neighborhood of c.*

Then (4) admits at least two positive T-periodic solutions.

Proof. We shall show that the validity of conditions H_1^* and H_2^* imply validity of $\tilde{H}_1$ and $\tilde{H}_2$ of Corollary 1. Given that $\frac{e^{-T}}{e^{-T}-1}\int_0^T f(s,y)ds$ strictly monotonically increasing with respect to y. Hence this monotonicity property holds for $\frac{1}{e^{-T}-1}\int_0^T f(s,y)ds$ also. If (a,c) represents a left neighborhood in which H_2^* is valid then by choosing c_1 to be any element of (a,c) we have $\max_{0\le y\le c_1}\left\{\frac{1}{e^{-T}-1}\int_0^T f(s,y)ds\right\}<c_1$. Thus $\tilde{H}_2$ is satisfied. From H_1^* and in view of H_2^* we have $\frac{e^{-T}}{e^{-T}-1}\int_0^T f(s,y)ds<y$ over (a,c) and there exists a $c_2>c$ such that $\frac{e^{-T}}{e^{-T}-1}\int_0^T f(s,y)ds\ge c_2$ for all $y\ge c_2$. This implies validity of $\tilde{H}_1$. Hence,

by Corollary 1, (4) admits at least two positive *T*-periodic solutions. ⊔

The Theorem 2, Corollary 1 and Theorem 3 involve conditions on integral of the function *f*(*t*,*y*). Below we present two existence theorems which are based on bounds of the function *f*(*t*,*y*).

Theorem 4. *Suppose that there exists two positive constant* $c_1<c_2$ *such that*

(H_3) $\min_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y(t))\right\}\ge\frac{c_2}{T}$ *when* $c_2\le y(t)\le\frac{c_2}{e^{-T}}$ *for all* $t\in[0,T]$,

(H_4) $\max_{\|y\|\le c_1}\left\{\max_{0\le t\le T}\left\{\frac{1}{e^{-T}-1}f(t,y(t))\right\}\right\}<\frac{c_1}{T}$.

Then (4) admits at least two positive T-periodic solutions.

Corollary 2. *Suppose that there exists two positive constant* $c_1<c_2$ *such that*

$(\tilde{H}_3)$ $\min_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\right\}=\frac{y}{T}$ *at* $y=c_2$ *and*

$$\min_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\right\}>\frac{c_2}{T}$$

for $c_2<y\le\frac{c_2}{e^{-T}}$

$(\tilde{H}_4)$ $\max_{0\le y\le c_1}\left\{\max_{0\le t\le T}\left\{\frac{1}{e^{-T}-1}f(t,y)\right\}\right\}<\frac{c_1}{T}$.

Then (4) admits at least two positive T-periodic solutions.

Theorem 5. *Assume that*

(H_3^*) $\min_{0\le t\le T}\left\{\frac{e^{-}}{e^{-T}-1}f(t,y)\right\}$ T *is strictly monotonically increasing function such that* $\lim_{y\to\infty}\min_{0\le t\le T}\left\{\frac{e^{-T}}{e^{-T}-1}\frac{f(t,y)}{y}\right\}=\infty$

(H_4^*) *there exists a* $c>0$ *such that* $\max_{0\le t\le T}\left\{\frac{1}{e^{-T}-1}f(t,y)\right\}<\frac{c}{T}$ *in a small left neighborhood of c.*

Then (4) admits at least two positive T-periodic solutions.

Proofs of Theorem 4, Corollary 2 and Theorem 5 are parallel to that of Theorem 2, Corollary 1 and Theorem 3 respectively. Hence the proofs are omitted.

4. Application to renewable resource dynamics involving additive Allee effects

In this section, we shall apply the results developed in the previous section to find the existence of two positive periodic solutions of (3). Let us consider the following form of (3).

(12) $$\frac{dy}{dt} = y - \frac{y^2}{\kappa(t)} - \frac{\eta(t)y}{1+m(t)y}.$$

In view of (4) we have

(13) $$f(t,y) = -\frac{y^2}{\kappa(t)} - \frac{\eta(t)y}{1+m(t)y}$$

where the coefficients $\eta(t)$ and $m(t)$ are positive T-periodic continuous functions. We have

$$\lim_{y\to\infty} \frac{e^{-T}}{e^{-T}-1}\frac{1}{y}\int_0^T f(s,y)ds = \lim_{y\to\infty}\frac{e^{-T}}{e^{-T}-1}$$

$$= \lim_{y\to\infty}\frac{e^{-T}}{1-e^{-T}}$$

.

Clearly H_1^* of Theorem 3 is satisfied by (3). We have the following theorem.

Lemma 3. *If there exists a $l \geq 0$ such that*

(14)
$$\int_0^T \frac{\eta(s)}{1+m(s)l}ds < 1 - e^{-T} - l\int_0^T \frac{ds}{\kappa(s)}$$

then there is a $c > l$ that satisfies H_2^*. Further, (3) admits at least two positive T-periodic solutions.

Proof. Let us consider the equation

(15)

$$\frac{1}{e^{-T}-1}\int_0^T\left(-\frac{y^2}{\kappa(s)} - \frac{\eta(s)y}{1+m(s)y}\right)ds = y.$$

Simplifying (15) we obtain

(16)
$$\int_0^T \frac{\eta(s)}{1+m(s)y}ds = 1 - e^{-T} - y\int_0^T\frac{ds}{\kappa(s)}.$$

Let us denote

$$r(y) = \int_0^T \frac{\eta(s)}{1+m(s)y}ds,\ s(y) = 1-e^{-T}-y\int_0^T\frac{ds}{\kappa(s)}$$ and

$$z(y) = \int_0^T\left(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\right)ds$$

defined on positive y axis. Observe that $z(y)$ is continuous and strictly increasing function, $r(y)$ is a convex function and it approaches zero as y approaches ∞. On the other hand $s(y)$ is a linear decreasing function intersecting the y axis at $(1 - e^{-T})/\int_0^T \frac{ds}{\kappa(s)}$. Let us assume that there exists $l \geq 0$ such that $\int_0^T \frac{\eta(s)}{1+m(s)l}ds < 1-e^{-T}-l\int_0^T\frac{ds}{\kappa(s)}$. In view of the qualitative behaviour of the functions $r(y)$ and $s(y)$, there exists a $c \in \left(l,(1-e^{-T})/\int_0^T\frac{ds}{\kappa(s)}\right)$ such that $r(c) = s(c)$. Since $z(y) < z(c)$ for $y < c$ we have

$$\int_0^T\left(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\right)ds < \int_0^T\left(\frac{c^2}{\kappa(s)} + \frac{\eta(s)c}{1+m(s)c}\right)ds$$

$$= c(1-e^{-T}) \text{ (since } r(c) = s(c)).$$

Since $z(y) = \int_0^T\left(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\right)ds$ is strictly monotonically increasing continuous function we obtain

$$\int_0^T\left(\frac{y^2}{\kappa(s)} + \frac{\eta(s)y}{1+m(s)y}\right)ds < c(1-e^{-T}).$$

The above inequality can be rewritten as

(17)

$$\frac{1}{e^{-T}-1}\int_0^T\Big(-\frac{y^2}{\kappa(s)}-\frac{\eta(s)y}{1+m(s)y}\Big)ds < c.$$

Therefore H_2^* of Theorem 3 is satisfied. Hence, from Theorem 3, (3) admits at least two positive *T*−periodic solutions. □

Now we shall apply Theorem 5 to investigate the existence of positive periodic solutions for (3). Since the coefficient functions $m(t),\eta(t)$ and $\kappa(t)$ are assumed to be positive periodic functions, there exists positive constants a,b,d,f,g and h satisfying

(18) $a \le \eta(t) \le b, d \le m(t) \le f$ and $g \le \kappa(t) \le h$.

In view of (18) we have

(19) $$\frac{y^2}{h}+\frac{ay}{1+yf} \le \max_{0\le t\le T}\Big\{\frac{y^2}{\kappa(t)}+\frac{\eta(t)y}{1+ym(t)}\Big\} \le \frac{y^2}{g}+\frac{by}{1+y}$$

.

Therefore we have

$$\begin{aligned}\lim_{y\to\infty}\min_{0\le t\le T}\frac{1}{y}\Big\{\frac{e^{-T}}{e^{-T}-1}f(t,y)\Big\} &= \lim_{y\to\infty}\min_{0\le t\le T}\frac{1}{y}\Big\{\frac{e^{-T}}{e^{-T}-1}\Big(-\frac{y^2}{\kappa(t)}-\frac{\eta(t)y}{1+m(t)y}\Big)\Big\}\\ &= \lim_{y\to\infty}\min_{0\le t\le T}\Big\{\frac{e^{-T}}{1-e^{-T}}\Big(\frac{y}{\kappa(t)}+\frac{\eta(t)}{1+m(t)y}\Big)\Big\} = \infty\end{aligned}$$

.

Therefore H_3^* of Theorem 5 is satisfied. We have the following Lemma, proof of which can be constructed parallel to that of Lemma 3. Lemma 4. *If there exists* $l' \ge 0$ *such that*

(20) $$\frac{b}{1+l'd} < \frac{1-e^{-T}}{T} - \frac{l'}{g}$$

then there is a $c > l'$ *that satisfies* H_4^*. *Further, (3) admits at least two positive T−periodic solutions.*

5. Discussion and Conclusions

Allee effects occur when ever fitness of an individual in a small or sparse population decreases as the population size or density also declines [6, 11, 12, 31]. The additive Allee effect that is known to occur when a prey dynamics is influenced by predator satiation [7, 34], group defence in a prey species, inhibition in micro organisms [1, 2] or difficulty in searching for a mate [34]. In all the works that concerned the additive Allee effects, the involved parameters have been taken to be constants implying that the dynamics are time independent or the environment is constant in time. But in natural world a biological organism's physical environment is non constant in time. Often the environment is either periodic or almost periodic. Periodicity in the environment is incorporated into the dynamics of a species by assuming that the involved coefficients in the equation governing its dynamics to be periodic [8, 13, 30]. In this article we have considered dynamics of a renewable resource that is subjected to additive Allee effect in a periodically varying environment. We have observed that, under reasonable conditions on the coefficient functions, there are at least two positive periodic solutions for the considered model.

The existence of at least two positive periodic solutions is obtained by employing *Leggett-Williams Multiple fixed point theorem* to the considered model. The existence is established using two different conditions. The first type involves integral conditions while the other involves bounds of the periodic coefficients. The existence results are illustrated through numerical simulation in section 5 using a suitable example. The theoretical results developed in this article provide an upper bound on the number of positive periodic solutions admitted by the considered model. It would be more interesting to obtain results that decide the exact number of periodic solutions along with their stability nature. Work in this direction is in progress.

REFERENCES

[1] Aguirre, P., Olivares, E.G., Saez,E., Two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect. *Nonlinear Anal. Real World Appl.*, **10**(2009), 1401 – 1416.

[2] Aguirre, P., Olivares, E.G., Saez, E., Three limit cycles in a Leslie-Gower predatorprey model with additive Allee effect, *Siam J. Appl. Math.*, **69**(2009), # 5, 1244 – 1262.

[3] Allee, W.C. *Animal aggregations*, University of Chicago Press, Chicage, IL, 1931.

[4] Allee, W.C. *Cooperation among animals*, Henry Schuman, New York,1951.

[5] Anderson, L.G., Seijo, J.C. *Bio-economics of Fisheries Management*, WileyBlackwell publishing, Singapore, 2010.

[6] Berec,L., Angulo, E., Counchamp, F., Multiple Allee effects and population management, *Trends Ecol. Evol.*, **22**(2006), # 4, 185 – 191.

[7] Boukal, D.S., Berec, L., Single-species models of the Allee effect:extinction boundaries, sex ratios and mate encounters, *J. theor. Biol.*, **218**(2002), 375 – 394.

[8] C´esar Castilho, Srinivasu, P.D.N., Bio-economics of a renewable resource in a seasonally varying environment, *Math. Biosc.*, **205**(2007), 1 – 18.

[9] Cheng, S.S., Zhang, G., Existence of positive periodic solutions for non-autonomous functional differential equations, *Electron. J. Differ. Equ.*, **59**(2001), 1 – 8.

[10] Clark, C.W., *Mathematical Bioeconomic: The optimal management of renewable resources (second edition)*, John Wiley and Sons, 1990.

[11] Courchamp,F., Brock, T.C.,Grenfell, B., Inverse dependence and the Alle effect, *Trends Ecol. Evol.*, **14**(1999), # 10, 405 – 410.

[12] Courchamp, F., Berec,L., Gascoigne,J., *Allee Effects in Ecology and Conservation*, Oxford University Press, Oxford, 2008.

[13] Cushing, J.M., Oscillatory population growth in periodic environments, *Theor. Popul. Biol.*, **30**(1986), 289 – 308.

[14] Dennis,B., Patil, G.P., The gamma distribution and weighted multimodel gamma distributions as models of population abundance, *Math. Biosci.*, **68**(1984), 187 – 212.

[15] Dennis, B., Allee effects: population growth, critical density, and the chance of extinction, *Nat. Resour. Model.*, **3**(1989),# 4, 481 – 538.

[16] Freedman, H.I., Wolkowicz, G.S.K., Predator-prey systems with group defence: The paradox of enrichment revisted, *Bull. Math. Biol.*, **8**(1986), 493 – 508.

[17] Jiang, D., Wei,J., Jhang, B., Positive periodic solutions of functional differential equations and population models, *Electron. J. Differ. Equ.*, **71**(2002), 1 – 13.

[18] Jin, Z.L., Wang, H., A note on positive periodic solutions of delayed differential equations, *Appl. Math. Let.*, *23*(2010), 581–584.

[19] Kent, A., Doncaster, C.P., Sluckin, T., Consequences for depredators of rescue and Allee effects on prey, *Ecol. Model.*, **162**(2003), 233 – 245.

[20] Leggett, R.W., Williamms, L.R., Multiple positive fixed points of non linear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**(1979), 673 – 688.

[21] Liermann, M., Hilborn, R., Depensation: evidence models and implications, *Fish and Fisheries*, **2**(2001), 33 – 58.

[22] McCarthy, M.A., The Allee effect, finding mates and theoretical models, *Ecol. Model.*, **103**(1997), 99 – 102.

[23] O'Regan, D., Wang, H., Positive periodic solutions of systems of first order ordinary differential equations, *Result. Math.*, **48**(2005), 310–325.

[24] O'Regan, D.,Wang, H., Positive periodic solutions of systems of second order ordinary differential equations, *Positivity*, **10**(2006), 285–298.

[25] Padhi, S., Srivastava, S., Multiple periodic solutions for non linear first order functional differential equations with applications to population dynamics, *Appl. Math and comp.*, textbf203(2008), 1 – 6.

[26] Padhi, S., Srivastava, S., Dix, J.G., Existence of Three Nonnegative Periodic Solutions for Functional Differential Equations and Applications to Hematopoiesis, Pan Amer. Math. J., 19(2009),# 1, 27 – 36.

[27] Pianka,E.R., Evolution ecology, New York: Harper and Row, 1978.

[28] Rojas-Palma, A., Gonzalez-Olivares, E., Betsabe Gonzalez-yanez, Metastability in a gause type predator-prey models with sigmoid functional response and multiplicative Allee effect on prey, In R. Mondaini and R.Dilao (Eds) Proccedings of the sixth Brazilian Symposium on Mathematical and computational Biology, Epapers Servicos Editoriais Ltda, Rio de Janeiro, 2006.

[29] Ruan, S., Xiao, D., Global analysis in a predator-prey system with nonmonotonic functional response, Siam J. Appl. Math., 61(2001), 1445 – 1472.

[30] Padhi, S., Srinivasu, P.D.N., Kiran Kumar, G., Periodic Solutions for an equation governing dynamics of a Renewable Resource Subjected to Allee Effects, Nonlinear Anal. Real World Appl., 11(2010), 2610 – 2618.

[31] Stephens, P.A., Sutherland, W.J., Consequences of the Allee effect for behavior, ecology and conservation, Trends Ecol. Evol., 14(1999), # 10, 401 – 405.

[32] Stephens, P.A., Sutherland, W.J., Vertebrate mating, Allee effects and conservation, World scientific Publishing, London, 2000.

[33] Taylor, R.J., Predation, Chapman and Hall, 1984.

[34] Thiem, H.R., Mathematics in Population Biology, Princeton University press, Princeton and Oxford, 2003.

[35] Wan, A., Jiang, D., Existence of positive periodic solutions for functional differential equations, Kvshu J.Math., 56(2002), 193 – 202.

[36] Wan, A., Jiang, D., A new existence theory for positive periodic solutions to functional differential equations, Comput. Math. Appl., 47(2004), 1257 – 1262.

[37] Wang, G., Liang ,X.G., Wang, F.G., The competitive dynamics of populations subject to an Allee effect, Ecol. Model., 124(1999), 183 – 192.

[38] Wang, H., Positive periodic solutions of functional differential equations, J. Diff. Equ., 202(2004), 354–366.

[39] Wang, H., Positive periodic solutions of singular systems with a parameter, J. Diff. Equ., 249(2010), 2986–3002.

[40] Wang, H., Positive periodic solutions of singular systems of first order ordinary differential equations, Appl. Math. Compu., 218(2011), 1605–1610.