BASICS OF MENGER SPACE AND THEOREMS RELATED TO MAPPING WITH CONTRACTION IN MENGER SPACES

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Abstract: The intent of this paper is to various issues triangular norm * Metric space, probabilistic metric space (PM – Space), Menger Space, probabilistic semi – metric space (PSM-space), various kind of contractions, self mappings and related theorems on different types of contractions, t-norm, common fixed point, PM spaces.

Keywords*:* t-norm, Menger spaces, Metric spaces, PM-spaces, PSM-spaces, contractions**,** Self mapping.

1. Introduction

This paper contains the notion of metric space is always plays a fundamental role, since continuity in analysis for real and complex functions, depends upon the notion of distance and the generalization of analysis totally depends upon to study the continuity. Metric space was introduced by French mathematician M. Frechet, since then there corresponds several generalizations of metric space in the literature the concept of alternation distance function has been used by many authors in a number of works. Analysis is an important branch of mathematics that was created in the seventeen century during scientific revolution. Functional analysis appears as a rather complex blend of Algebra and Topology. It does not have only great influence in mathematics but both pure and applied mathematicians are dealing with the nonlinear equations, which have great influence in physical, social, engineering and different technological fields. Analysis has been classified into two major branches: linear and non-linear analysis. It is a probabilistic generalization in which assign to any two points x and y, a distribution function G_{xy}studied this concept and gave some fundamental results on this space.

The purpose of this paper is to study briefly on Menger spaces and various kinds of contractions, self mappings, and related theorems on different types of contractions, t-norm, metric space and its associated properties with some fixed point results.

2. Preliminaries

Definition 2.1: A triangular norm * is a binary operation on the unit interval [0,1] suchthat for all $a, b, c, d \in [0,1]$ the following conditions are satisfied.

i) $a * 1 = a$
ii) $a * b = b * a$
iii) $a * b \leq c * d$ whenever $a \leq c \text{ and } b \leq d$
iv) $a * (b * c) = (a * b) * c$

Triangular norm is often known as t-norm and also t-norm is a function

$G: [0,1] \times [0,1] \to [0,1]$which is associative,commutative, non decreasing in eachcoordinate and $G(a, 1) = a$for all $a \in [0,1]$

Examples of t-norms are

i) $a * b = min\{a, b\}$

ii) $a * b = max\{a + b - 1,0\}$

iii) $a * b = a \times b$

And also a t-norm is a 2-place function.

Definition 2.2: A distribution function is a function $G: [-\infty, \infty] \to [0,1]$

Which is left continuous on R, non-decreasing and $G(-\infty) = 0, G(\infty) = 1$

We will denote by Δ the family of all distribution function on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0 \; if \; t \le 0 \\ 1 \; if \; t > 0 \end{cases}$$

If X is a nonempty set, $G: X \times X \to \Delta$ is called a probabilistic metric space (PM – Space) if X is a nonempty set and G is a probabilistic distance satisfying the following conditions:

for all $x, y, z \in X \; and \; t, x > 0$.

i) $G_{xy}(t) = 1 \;\; \Leftrightarrow x = y$
ii) $G_{xy}(0) = 0$
iii) $G_{xy} = G_{yx}$
iv) $G_{xy}(t) = 1, G_{xy}(s) = 1 \Rightarrow G_{xy}(t+s) = 1$

The ordered triple $(X, G,*)$ is called Menger space if (X, G) is a PM- Space, * is a t-norm and the following condition is also satisfies

for all $x, y, z \in X \; and \; t, s > 0$

$$G_{xy}(t+s) \ge G_{xy}(t) * G_{zy}(s)$$

If only (i), (ii) and (iii) hold, then (X, G) is said to be a probabilistic semi – metric space (PSM-space). We note that every symmetric(semi-metric)space (X, d) can be realized as a probabilistic semi - metric space by taking

$G : X \times X \to \Delta$defined by $G_{xy}(t) = H(t - d(x,y)) \forall \; x, y \in X.$

For every PSM- space (X, G) we can consider the sets of the form

$$U_{\in,\lambda} = \{(s,y) \in X \times X / F_{xy}(\in) > 1 - \lambda\}$$

The family$\{U_{\in,\lambda}\}_{\in>0, \lambda\in(0,1)}$ generates a semi uniformity denoted by$U_G$and a topology$\mathcal{T}_G$ called the F-topology or the strong topology. Namely,

$$A \in \mathcal{T}_G \; if \; \forall x \in A \exists \in > 0 \; and \; \lambda \in (0,1) \ni \; U_{\in\lambda} \subset A. U_G$$

Is also generated by the gamily $\{V_\delta\}_{\delta>0}$ where $V_\delta := V_{\delta,\delta}$

In it is proved if $sup_{t>0}(t * t) = 1$ then U_G is uniformity, called F-uniformity which is metrizable.The G-topology is generated by G-uniformity and is determined by the G-convergence:$x_n \to \Leftrightarrow G_{x,}x(t) \longrightarrow 1 \forall t > 0$.

Definition 2.3: A t-norm * is continuous if for all convergence sequences $\{x_n\}_{n\in N} and \; \{y_n\}_{n\in N}$

We have $\lim_{n\to\infty} x_n * \lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_n * y_n)$

Definition 2.4: Let $(X, G,*)$ be a Menger space and * be a continuous t-norm.

A sequence$\{x_n\}$ in X is said to be converge to point in iff for every$\in > 0 \; and \; \lambda \in (0,1)$ and there exists an integer $n_0 = n_0(\in, \lambda)$ such that$G_{xnx}(\in) > 1 - \lambda, \forall n \ge n_0$

A sequence$\{x_n\}$ in X is said to be Cauchy if for every$\in > 0 \; and \; \lambda \in (0,1)$ and there exists an integer $n_0 = n_0(\in, \lambda)$ such that$G_{x_{n=p}}(\in) > 1 - \lambda, \forall n \ge n_0 and \; p > 0$

A Menger Space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5: A topological space is called separable if it contains a countable dense subset, that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such thatevery nonempty open subset of the space contains at least one element of the sequence.

Definition 2.6: Let (X, G, *) be a Menger space and * be a continuous t- norm.Then, a sequence $\{x_n\}$in a Menger space (X, G,*) is said to converge to a point x inX, if for every $\in > 0$ and $\lambda\epsilon(0,1)$ there is an integer $n_0 = n_0(\epsilon, \lambda)$such that$G_X, x_n(\varepsilon) > 1 - \lambda, \forall n \ge n_0$.It is noted that the limit of a convergent sequence in

PM-space is always unique.

Definition 2.7: A PM space X is said to be compact if every sequence in X has a
convergent subsequence.

Definition 2.8: Let $f: X \to X$ be a mapping. A point x in X is a fixed point of f if, $f(x) = x$.

Definition 2.9: Let M be a set and the distance function d is a mapping from $M \times M$ into set of non-negative real numbers(R^+), satisfying the following conditions

i. *d(x, y) ≥ 0* for x ≠ y, (Non-Negativity)
ii. *d(x, y) = 0* iff x = y, (Identityof Indiscernibles)
iii. *d(x, y) = d(y, x)*, (Symmetricity),
iv. *d(x, z) ≤ d(x, y) + d(y, z)*, (Subaddivity/Triangular Inequality),
v. *d(x, z) ≤ d(x, y) ∪ d(y, z)*, (Strong Triangular Inequality), and
vi. *d(x, x) = 0*, (Reflexivity).

Then, the pair (M, d) is called

1. Metric space if it satisfies conditions (i), (ii), (iii)

and (iv).

2. Pseudometric space if it satisfies conditions (i), (iii), (iv) and (vi).

3. Quasi-metric space if it satisfies conditions (i), (ii), and (iv), and

4. Semi- metric space if it satisfies conditions (i), (ii) and (iii).

Definition 2.10: Let (X, G)be a PM space. A mapping $T: X \to X$is a contraction mapping on (X, G)if and only if there exists α belongs to (0,1)such that $G_{T_p,T_q}(t) \geq G_{p,q}(t/\alpha)$where $p, q \in X$ and $t > o$. And a self mapping f on a PSM space is called B-Contraction, if there is a $\gamma \epsilon (0,1)$such that for all points p,q in S and for all

x>0 $G_{f_p,f_q}(t) \geq G_{p,q}(x)$.

Definition 2.11: Let (S, G) be a PM space. A mapping $T: S \to S$is called H-contraction.

If there exists k belongs to (0,1) such that $G_{f_p,f_q}(kx) > 1 - kt$

whenever $G_{p,q}(x) > 1 - t$, where $p, q \in X$ &$t > o$.

Definoition2.12: Let f, g are the self mapping on a metric space (X, d) are said to be weakly commutative if $d(fgx, gfx) \leq d(fx, gx) \forall x \in X$.

Definition 2.13: Let f, g are the self mapping on a metric space(X, d) are said to be weakly compatible if whenever $\{x_n\}$ is a sequence in X such that both $\{fx_n\}\{gx_n\}$ are convergent to a same point x in X then $d(fgx_n, gfx_n) \to 0$.

Definition 2.14: Every metric space (M, d) is a menger space (M, G, min)here the mapping $G(x, y) = G_{x,y}$ is defined by $G_{x,y}(\varepsilon) = H(\varepsilon - d(x, y))$, and H is the distribution function defined by $H(\varepsilon) = \begin{cases} 0 \ if \ \varepsilon \leq 0 \\ 1 \ if \ \varepsilon > 0 \end{cases}$

The space (M, G, min) is called the induced Menger space.

3. Main results

Theorem 3.1: The mapping $f: S \to S$ is an H-contraction on the PM space

(S, G, τ)With$\tau \geq \tau M$if and only if f is a contraction on the metric space$(S, \beta)$with this Condition$\tau \geq \tau M$, H- contraction on complete PM space has a fixed point, but it is not true for B-contraction. So, it is clear that B-contraction need not be H-contraction. Also, H-contraction need not be B-contraction.

Theorem 3.2: If f is B-contraction on the PSM space(S, G) and if the distributionFunctionG_{f_p,f_q} is strictly increasing on [0, 1], then$\beta(f_p, f_q) < \beta(p, q)$.

In this section, we state some fixed point results for single self maps in PM and PSM spaces.

Theorem 3.3: A sequence $\{p_n\}$ in a PM space X converges to a point p if and only if for every $x \epsilon R$ we have lim $F_{p,p_n}(x) = H(x)$.

Theorem 3.4: Let (E, G, Δ)be a complete PM space with a t-norm Δ satisfying $\Delta(\mathrm{t}, \mathrm{t} \geq \mathrm{t})$for all $t \epsilon [0,1]$.

Let T: E $\rightarrow$ E be a mapping with $G_{T_p,T_q}(t) \geq G_{p,q}(t/k(\alpha,\beta))$ for all $p,q \in X$ and $t \geq 0$.and $\alpha,\beta \in (0,+\infty)$with $G_{x,y}(\alpha) > 0$ and $G_{x,y} < 1$where

$k(\alpha,\beta): (0,+\infty)^2 \rightarrow (0,1)$is a function, then T has exactly one fixed point in E. In the sequel the mapping T that has satisfied above equation is called generalized contraction mapping on E.

Theorem 3.5: Let $(X,G)$is a PM space and$\alpha: [\alpha,\beta) \rightarrow [0,\infty)$ is strictly increasing and

satisfying$\alpha(0) = 0$ and $\alpha(t) < t\ \forall t > 0$.If $x,y$are two members in X such that

$F_{x,y}(\alpha(\varepsilon)) \geq F_{x,y}(\varepsilon)and\forall \varepsilon > 0$, then $x = y$

Theorem 3.6: Suppose $(M,d)$is a complete metric space and S,T,A,B are four self maps on M satisfying the following

(i) $SM \subseteq BM$and$TM \subseteq AM$

(ii)$(S,A)$and$(T,B)$are compatible pairs

(iii) One of S, T, A and B is continuous

(iv) There exists an upper semicontinuous function $\pi: [\alpha,\beta) \rightarrow [0,\infty)$ with $\pi(0) = 0$and$\pi(t) < t$, for $t > 0$such that for $\varepsilon > 0$and$x,y \in M$, if $\pi(\varepsilon) \leq d(Sx,Ty)$then either

$\varepsilon \leq \max\{d(Ax,Sx), d(By,Ty)$ or

$2\varepsilon \leq \max\{d(Ax,Ty), d(By,Sx)\}$.

Then S, T, A and B have a unique common fixed point.

Proof: Assume that for any $\varepsilon > 0$ and for any$x,y \in M$if $\varepsilon > \max\{d(Ax,Sx), d(By,Ty)$ then $\pi(\varepsilon) > d(Sx,Ty)$. We now that for any $x,y \in M$and any $\varepsilon > 0$Indeed.

Let x,y be any two points in M and ε be any positive number.

In case $\varepsilon > \max\{d(Ax,Sx), d(By,Ty)$

Then we have $(H(\varepsilon - d(Ax,Sx) = 1 = H(\varepsilon - d(By,Ty)and\pi(\varepsilon) > d(Sx,Ty)$.

So the following inequalities are hold.

Theorem 3.7: Let $(X,G,*)$be a complete Menger space with$t * t \geq t\ \forall\ t \in [0,1]$ and let A and L be compatible maps on X such that$L(X) \subseteq A(X)$. If A is continuous and there exists a constant$k \in (0,1)$such that

$$G^2_{LxLy}(kt) * [G_{AxLx}(kt).G_{AyLy}(kt)]$$

$$\geq pG_{AxLx}(t) + qG_{AxLx}(t)].G_{AxLy}(2kt)]$$

For all $x,y \in X$ and $t > 0$where$0 < p,q < 1$such that $p + q = 1$then A and L have a unique fixed point.

Theorem 3.8: Let (X,d) be a metric space, and let $T: X \rightarrow X$be a selfmapping on X.

Define$G: X \times X \rightarrow L$ by

$$[G(p,q)](x) = G_{p,q}(x) = H(x - d(p,q))$$

$\forall\ p,q \in X$and $x \in R$, where $\{G_{p,q}: p,q \in X\} \subseteq L$.

Example: Suppose that t-norm $t: [0,1] \times [0,1] \rightarrow [0,1]$is defined by $t(a,b) = \min\{a,b\}$for all$a,b \in [0,1]$Then,

i. (X,G,t) is a Menger space

ii. If (X,d) is T orbitally complete, then (X,G,t) is T orbitally complete.

Note: Menger space generated by a metric is called the induced Menger space.

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