Existence and Stability of exactly two positive periodic solutions for an equation governing dynamics of weak Allee Effect

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Abstract: Allee effect plays an important role on the evolution of population in ecological systems. One important concept is the Allee threshold that determines the persistence or extinction of the population in a long time. In general, a small initial population size is harmful to the survival of a species since when the initial data is below the Allee threshold the population tends to extinction, rather than persistence. Another interesting feature of population evolution is that a species whose movement strategy follows a conditional dispersal strategy is more likely to persist. The coexistence of the above two conflicting mechanisms makes the dynamics rather intricate. However, some numerical results obtained by Cosner et. al. show that the directed movement can invalidate the strong Allee effect and help the population survive. To study this intriguing phenomenon, we consider the pattern formation and local dynamics for a class of single species population models of that is subject to the strong Allee effect. We first rigorously show the existence of multiple localized solutions when the directed movement is strong enough. Next, the spectrum analysis of the associated linear eigenvalue problem is established and used to investigate the stability properties of these interior spikes. This analysis proves that there exists not only unstable but also linear stable steady states. Then, we extend results of the single equation to coupled systems, and also construct several non-constant steady states and analyze their stability. Finally, numerical simulations are performed to illustrate the theoretical results.

1 Introduction

In this paper, we mainly investigate the aggregation phenomena and dynamics of the following single reaction-diffusion equation for $u = u(x,t)$ with the no-flux boundary condition:

$$u_t = \nabla \cdot (d_1 \nabla u - \chi u \nabla A) + \mu u(1 - u)(u - \theta),$$
$$(d_1 \nabla u - \chi u \nabla A) \cdot \mathrm{n} = 0, \; u(x,0) = u_0(x) \geq 0,$$

Here x and t are space and time variables, d_1, χ and μ are arbitrary positive constants, and n is the unit outer normal on the boundary $\partial\Omega$. To construct the non-constant steady states, we also focus on the stationary problem of (1.1), which is:

$$0 = \nabla \cdot (d_1 \nabla u - \chi u \nabla A) + \mu u(1 - u)(u - \theta),$$

$$(d_1 \nabla u - \chi u \nabla A) \cdot \mathrm{n} = 0,$$

Equation (1.1) serves as a paradigm to describe the dynamics of one population with the effect of some known signal subject to the Allee Principle [1,30], where $u : \Omega \times [0, \infty) \to [0, \infty)$ denotes the density of a population and A is a known stimulus that governs the directed movement; the constant d_1 represents the population diffusion rate, χ reflects the strength of the biased movement, while the source $f(u) := u(1 - u)(u - \theta)$ models the Allee effect and $\theta \in (0,1)$ is the Allee threshold.

The general form of system (1.1) was proposed by Cosner and Rodriguez [11], which reads:

where $B[u] = 0$ represents either homogeneous Dirichlet or no-flux boundary conditions. In particular, they obtain a set of qualitative and numerical results concerning the short time dynamics and steady states of system (1.3). Moreover, to study the interaction between two species, they extended equation (1.3) to the following system:

$x \in \Omega, t > 0, x \in \partial\Omega, t > 0, x \in \Omega.$ (1.1)

$$u_t = \mathrm{M}_u u + ug(x, u + v), \qquad (1.4)$$
$$v_t = \mathrm{M}_v v + vg(x, u + v),$$

where (u,v) are the population densities of two species and the dispersal operators are defined by

$$\mathrm{M}_u u := \nabla \cdot (\nabla u - \chi_1 u \nabla A), \quad \chi_1 > 0,$$

and $x \in \Omega$, $x \in \partial\Omega$. (1.2)

$$\mathrm{M}_v v := \nabla \cdot (\nabla v - \chi_2 v \nabla A), \quad \chi_2 > 0;$$

while the growth pattern is $g(u + v) := (r - u - v)(u - v - \theta)$ and where r is the given resources. Some numerical results for (1.4) presented in [11] demonstrated that two populations cooperate at low densities and compete at high densities.

To study this phenomenon, we consider the coupled system (1.4) in the following two cases of χ_1 and χ_2:

(i). $\chi_1 = \chi$, $\chi_2 = 1$, where $\chi > 0$ represents the speed of the intra-species;

(ii). $\chi_1 = \chi$, $\chi_2 = c\chi$, where constant $c > 1$ implies the inter-species is faster.

In particular, we prove the existence of non-constant steady states for system (1.4) in case (i) and case (ii), then study their stability properties.

1.1 *Allee Effect*

The well-accepted definition of Allee effect is the positive relationship between population density and individual fitness. This effect often occurs under situations involving the survival and reproduction of animals, such as habitat alteration, mate-finding [13, 16], etc.

In terms of the scale, the Allee principle is typically decomposed into the component Allee effect and the demographic Allee effect. The former emphasizes the relationship between any measurable component of survival rates and density size [3], while the latter highlights the overall correlation between them [23]. Many researchers tend to consider macro-population problems, and thereby the demographic Allee effect is more popular. Some significant concept therein is the critical population size. When a population threshold exists, the demographic Allee effect is the so-called strong Allee effect; otherwise it is named the weak Allee effect. In general, when the initial density is below (above) the critical threshold, the population tends to be extinct (persistent). The critical population size is called the Allee threshold and the relevant models have been intensively studied, see [25, 31– 33].

The most popular and simplest equation used to model the population dynamics subject to the strong Allee effect is

$$u_t = u(r - u)(u - \theta),$$

where r represents the environmental resources and $\theta \in (0,1)$ is the Allee threshold. Here we define $g(u) := (r-u)(u-\theta)$ which admits the bistable growth pattern. It can be seen that when the environment is homogeneous, $u \equiv \theta$ and $u \equiv r$ are two constant equilibria. In particular, $u \equiv \theta$ is unstable and $u \equiv r$ is stable.

1.2 *Directed Movement: Taxis and Advection*

A taxis is the mechanism by which organisms direct their movements in response to the environmental stimulus gradient. In terms of stimulus such as wind, light, chemical signal, etc., taxis can be identified as Anemotaxis, Phototaxis, Chemotaxis and so on. In particular, the effect of taxis on population dynamics is often interpreted as the conditional dispersal of species [26] and from the viewpoint of mathematical modelling, the advection term presents a paradigm to model it.

Combining the biased and unbiased dispersal, many reaction-diffusion-advection models have been proposed in the literature to analyze biological problems involving population dynamics. The survey paper [9] summarizes a class of such systems and their applications. The conditional dispersal in general is a benefit for the persistence of a species [2], with the sensible explanation that individuals can perceive the favorable environmental signals such as the presence of food, and then move towards the stimulus and finally aggregate.

There have been many previous results for the case where the population dynamics follows a logistic growth [2, 4, 6, 10, 19]. In particular, Belgacem and Cosner [2] considered the following reaction-diffusion-advection model:

$$\begin{aligned} &u_t = \nabla \cdot (d_1 \nabla u - \chi u \nabla A) + \mu u(A - u), \\ &\quad (d_1 \nabla u - \chi u \nabla A) \cdot \mathrm{n} = 0, x \in \Omega, t > 0, x \in \partial\Omega \\ &\quad u(x,0) = u_0(x) \geq 0, t > 0, x \in \Omega, \end{aligned}$$

where the environment is spatially heterogeneous and the boundary acts as a reflecting barrier. They proved the population tends to be persistent if χ is large, which implies that the strong advection effect is beneficial. Moreover, they shew that there exists some unique non-negative constant $\bar{\mu}_*$ depending on χ such that when $\mu > \bar{\mu}_*$, (1.5) admits a unique positive global attractor. Cosner and Lou [10] further showed that the effect of the biased movement is not always beneficial and depends crucially on the shape of the domain, where it was established that non-convex domains can be harmful to the persistence of the population. (See also interesting related results in Chen and Lou [8].) There are also many different results when the boundary condition is assumed to be Dirichlet:

$$u(x,t) = 0, \qquad x \in \partial\Omega, t > 0,$$

which is the so-called lethal boundary. For instance, a strong drift term may be harmful rather than helpful [2], and more interesting results were shown in [4, 20]. Similar to the logistic growth, Allee effects also have rich applications in modelling population dynamics. There are a few references focused on discussing the models subject to Allee effects [5, 27, 27].

1.3 *Ideal Free Distribution Strategy*

The ideal free distribution (IFD) was introduced by Fretwell [15] in 1969 to describe how one species distribute individuals to minimize competition and maximize fitness. The theory states that under the following assumptions:

(i). Individuals in the species are homogeneous and equally able to access resources;

(ii). Individuals are free to move in the environment;

(iii). Organisms understand how to acquire the largest amount of resources and maximize fitness,

the arrangement of individuals exactly matches the distribution of resources in the environment. In general, the external resources are supposed to be located at several sites and form various aggregates, then homogeneous individuals will move towards sites and distribute themselves among these patches of resources. More specifically, the number of individuals aggregated in each patch is proportional to the amount of available resources.

The IFD strategy can be modelled by the following equation:

$$v_t = \nabla \cdot (\nabla v - v\nabla \ln r) + v(r - v)(v - \theta),$$

where the external resources, modelled by r, are fixed. In this equation, one finds that $v = r$ is an equilibrium, which implies the distribution of the species is the same as that of resources. In this article, we study how the strategies including IFD strategy and aggressive strategy, influence the persistence of species.

1.4 *Motivations and Main Results*

Cosner and Rodriguez [11] combined the free and conditional dispersal to model the movement of a population with the assumption that its dynamics is governed by the strong Allee Principle. They proposed (1.3) and studied the existence of equilibrium subject to the lethal boundary and reflecting barrier. Furthermore, some numerical simulations were presented to illustrate that the biased movement plays a vital role on overcoming a strong Allee effect. The figures in [11] show if χ is large, i.e. the advection effect is strong, the population will persist rather than disappear even though the initial size is below the Allee threshold θ.

To confirm this numerical experimental finding, we perform theoretical studies by considering system (1.1) and (1.2). Our main goal is to rigorously construct non-constant solutions of (1.2), and then investigate their stability properties within (1.1). In particular, since we focus in understanding the influence of the conditional dispersal rate χ on the strong Allee effect, we set the remaining parameters d_1 and μ to one.

An immediate consequence of the no-flux boundary condition is the following integral constraint satisfied by all classical solutions of (1.2):

$$Zu(1 - u)(u - \theta)dx = 0. \qquad (1.6)$$

It can be observed from (1.6) that system (1.1) can admit different nontrivial patterns. Indeed, some formal analysis implies this integral constraint determines the height of each local interior spike. We suppose A is smooth and radial with only one non-degenerate local maximum point at 0. Then we expand A as $A = A_0 - \frac{a}{2}|x|^2 + O(|x|^3)$, where $A_0 := A(0)$ is the local maximum of A and $a := A_{rr}(0) > 0$. Set $U = U_0 + \epsilon U_1 + \cdots$ and let $\chi := \frac{1}{\epsilon^2}$, $y := \frac{x}{\epsilon}$, $U(y) := u(x)$, $F(U) := f(u)$ to obtain the following

leading order equation:

$$0 = \nabla_y \cdot (\nabla_y U_0 + aU_0 \cdot y), \quad U_0(y) \to 0, \ as \ |y| \to \infty,$$

$$F(U0)dy = 0. \qquad (1.7)$$

$$U0 = c0e - a2\ |y|2. \qquad (1.8)$$

Upon Substituting U0 into (1.6), we find c0 satisfies

$$2^2 c^2_0 - (1 + \theta)3^2 c_0 + 6^2 \theta = 0.$$

It is easy to check that there exists

$$\theta_1 := \frac{2^{n+1} - 2\sqrt{4^n - 2^n \cdot 3^{\frac{n}{2}}} - 3^{\frac{n}{2}}}{3^{\frac{n}{2}}} \in (0,$$

where $\delta := 3^n\theta^2 + 2 \cdot 3^n\theta - 4 \cdot 12^{\frac{n}{2}}\theta + 3^n$.
Thanks to (1.12) and (1.8), the asymptotic profiles of single interior spikes can be expressed explicitly and are shown in Figure 1.

We would like to point out that when $\theta \in (\theta_1,1)$, (1.7) only admits the solution $U_0 = 0$ since the quadratic equation (1.10) does not have any real solution. As a consequence, when $\theta \in (\theta_1,1)$, there only exists a trivial pattern or the only one non-trivial spatially homogeneous pattern to (1.1) what we are not interested in. Therefore, we only focus on the case $\theta \in (0,\theta_1)$ rather than $\theta \in [\theta_1,1)$. The above formal argument supports our claim that the height of a spike, given by c_0, is determined by the integral constraint (1.6) and given by (1.12). Moreover, our forthcoming rigorous analysis will prove that this statement holds for not only this special form for A but also for a more general class of A.

Before stating our main results for the pattern formation of (1.1), we discuss the properties of the signal A. Indeed, it plays the vital role for the formation of nontrivial patterns within (1.1). Numerical simulations exhibited in [11] show that the non-constant steady states to (1.1) tend to be concentrated at the local non-degenerate maximum points of A. The formal asymptotic analysis given above also confirms this fact. Now, we recall the assumptions satisfied by the admissible signal A in [11], which are as follows:

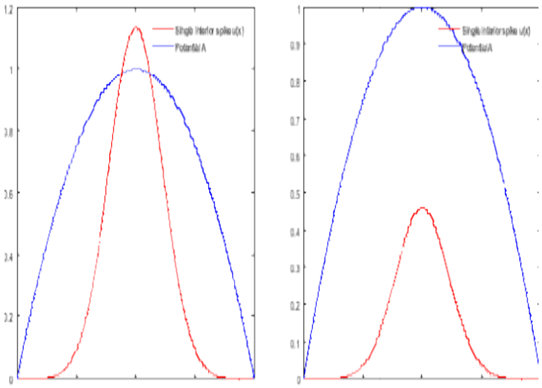


Figure 1: *For a 1-D domain with given potential* $A = 1 - x^2$, *Allee threshold* $\theta = 0.3$ *and conditional dispersal rate* $\chi = 10$, *we show the leading order profiles of single interior spikes defined by (1.8) with* $c_0 = c_{01} = 1.1339$ *(left) and* $c_0 = c_{02} = 0.4582$ *(right).* We shall prove that the left single interior spike is stable but the right one is unstable.

(A2). $\|\Delta A\|_{L^\infty(\Omega)} \leq M$ for some constant $M > 0$.

Assumption (A1) and (A2) are technical assumptions needed for the analysis. For our analysis below, we also propose several new hypotheses on A:

(H1). all critical points of A are either local non-degenerate maximum points, or critical points with

$\Delta A > 0$;

(H2). $\frac{\partial A}{\partial \mathbf{n}} < 0$ holds for all $x \in \partial\Omega$.

After supposing A admits exactly k non-degenerate local maximum points $x_1, \cdots, x_k$, where $x_m := x_m^{(1)}, \cdots, x_m^{(n)T}$, $m = 1, \cdots, k$. we have from assumption (A1), (A2) and hypothesis (H1), (H2) that A can be expanded at x_m as

$$A = Am - \frac{1}{2} \mathrm{X}\, x(i) - xmi\ T$$
$$h(mij)\, x(i) - xm(i)$$

where $A_m := A(x_m)$ and $-h^{ij}_m$ is the ij-th entry of the Hessian matrix of A at x_m. It is necessary to point out that the Hessian matrix of A at every local non-degenerate maximum point x_m is negative definite. where the notations $h^{(\ i)}_m$, $x^{(i)}$ and $x_m^{(i)}$ are used to substitute $\hat{h}^{(\ i)}_m$, $\hat{x}^{(i)}$ and $\hat{x}_m^{(i)}$ in (1.14), respectively without confusing readers. With the help of the above discussion, now we summarize the first set of our results regarding the stationary problem in the following theorem:

Theorem 1.1. *Under the assumptions (A1)-(A2) and hypotheses (H1)-(H2), for each fixed positive integer* l *satisfying* $1 \leq l \leq k$, *we have that there exists* χ_0 *such that when* $\chi > \chi_0$, *(1.2) admits a solution having the following form:*

$$\backslash \mathrm{X} k - \underline{1}\, \mathrm{P}^{n\ h}{}_m(i)\chi(x - {}^{x}{}_m(i))2$$

Remark 1.1. *As is shown in Theorem (1.1), in contrast to many reaction-diffusion-advection models such as the minimal Keller–Segel model [21] where the single interior spiky solution is unique [7], (1.2) has a variety of single interior spikes located at each local non-degenerate maximum point of A.*

Numerical Studies and Discusion

In this section, several set of numerical simulations are presented to illustrate and highlight our theoretical analysis. We apply the finite element method in FLEXPDE7 [14] to system (1.1) with the error is 10^{-4}. Besides supporting our theoretical results, our numerical simulations show that system (1.1) admits rich spatial-temporal dynamics.

Figure 3 and Figure 4 exhibit the pattern formation within system (1.1) when A has only one local non-degenerate maximum point. These figures illustrate that the single interior spike given by (1.15) with the height is c_{01} is linearly stable. Similarly, Figure 5 shows that the single interior spike defined in (1.15) with the other positive height is unstable and some small perturbation will cause the time-dependent solutions to (1.1) move away from it.

We next present the stability of multi-interior spikes defined in (1.15) when signal A admits two local non-degenerate maximum points. Before that, the asymptotic profiles of them are shown in Figure 6. Similar to the single interior spike, our numerical result shown in Figure 7 indicates that those multiple spiky solutions whose every bump has the larger height are local linearly stable. In contrast, once one of their bumps admits the smaller height, the stationary solutions will become unstable, as shown in Figure 8.

Figure 9 and Figure 10 exhibit the large time behavior of solutions to (4.1) and (4.2), respectively. From the viewpoint of population ecology, the numerical results shown in Figure 9 can be interpreted that the conservative species will be better off in the long run when the Allee threshold is small. This phenomenon is counter-intuitive since one might believe that the higher speed benefits the persistence of a species, so that an aggressive species is more likely to survive. Our result demonstrates that aggressive strategy is not always optimal and that an IFD strategy is preferable for species persistence in some cases. A further qualitative result shown in Figure 10 is that competitive species does not like to coexist and, instead, prefer to occupy all resources by themselves. Our interpretation of this result is that aggressive species do not want to share any resources with each other.

Discussion

We have used the reduction method to construct and study the linear stability of localized solutions to the single species models (1.1) and competition models (1.4) in the limit of an asymptotically large

speed $\chi \gg 1$. Our main contribution has been the rigorous analysis of the existence of localized patterns and their stability properties. Under the technical assumptions (A1), (A2) and (H1), (H2), we shew that (1.1) admits many localized solutions when the potential A has multiple maximum points. In particular, there are two possible heights for every local bump. Regarding the stability properties, we proved that once some local bump has the small height, the spike will be unstable. We next focused on the analysis of the population model (1.4). On the one hand, we proved the noncoexistence of intra-species and inter-species who follow the aggressive strategy and IFD strategy, respectively. Moreover, we found that when the Allee threshold $\theta = O(1)$, the species who follows the aggressive strategy will persist in the long run; while $\theta = O\frac{1}{\underline{n}}$, the aggressive strategy will lead

χ 2

to the extinction of species in some cases. On the other hand, with the assumption that two species both follow the aggressive strategy, we shew that even though the localized patterns might coexist in local bumps, they are unstable and the more aggressive species will persist in the long term.

We would like to mention that there are also some open problems that deserve future explorations. While discussing the existence of interior spike steady states, we impose some technical assumptions on A; for instance, we assume that A has k non-degenerate maximum points. Whether or not these

spikes in the rectangular area.

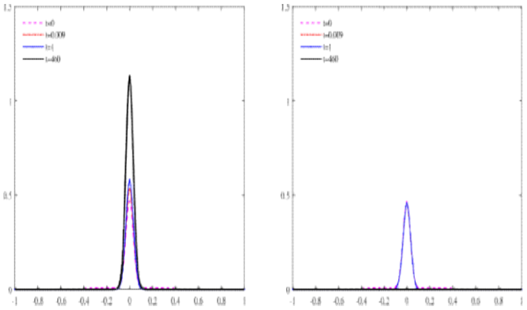


Figure 5: $\theta = 0.3$ *and* $\chi = 10$*. Left: the solution of system (1.1) in 1-D at* $t = 0$*, 0.009, 1 and* 460*s with initial data* $u_0(x) = 0.46e^{-50\chi x2} + 0.01\cos(2x)$*;*

Right: the solution of system (1.1) in 1-D with initial data $u_0(x) = 0.46e^{-50\chi x2} + 0.001\cos(2x)$. We have $u(x,t)$ at $t = 460$ can represent the steady state to (1.1) and it is shown that the single interior spike with the smaller height is unstable and converges to either the steady state with the larger height or zero in a long term.

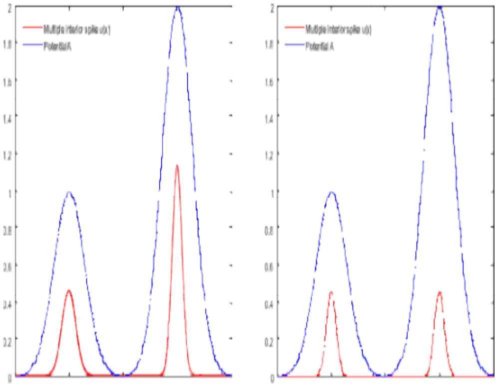
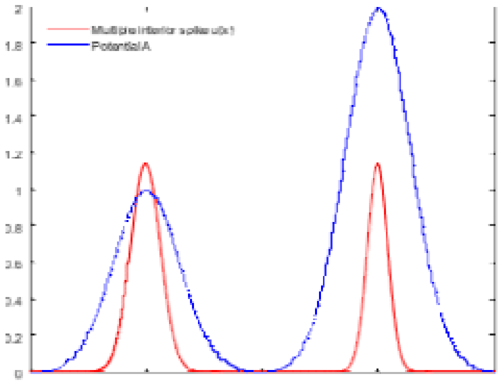
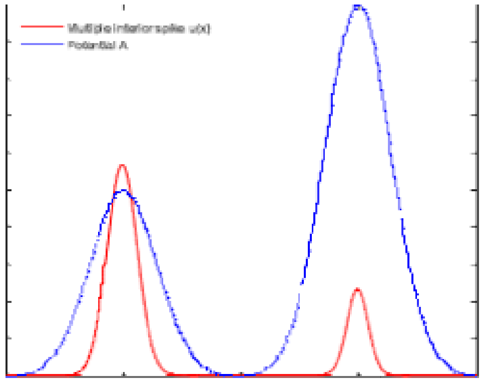


Figure 6: *For a 1-D domain, we have four types of interior spikes constructed in Theorem 1.1 with the height of each bump is either* $c_{01} = 1.1339$ *or* $c_{02} = 0.4582$ *when potential* $A = \frac{5}{\sqrt{2\pi}}e^{-25(x-0.5)^2} + \frac{5}{2\sqrt{2\pi}}e^{-25(x+0.5)^2}$, *Allee threshold* $\theta = 0.3$ *and conditional dispersal rate* $\chi = 10$. We find that (1.1) admits a variety of interior spikes when A has more than 1 local maximum points assumptions can be removed remains an open problem.





Besides the stable interior spikes, we believe that (1.1) also admits the stable boundary spikes, and the rigorous analysis needs to be established. Regarding the population system (1.4), we only study the influence of large advection on the population evolution of interacting species. The effect of small diffusion d_1 is apparently another delicate problem that deserves probing in the future.

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