Optimal Harvesting Polices for single species Models

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Abstract: In this article bio-economics of a renewable resource that is subjected to single species models are investigated from sole owner perspective. The considered optimal harvesting problem has been solved using Pontryagin maximum principle. The control problem admits multiple singular equilibrium solutions in contrast to the case where the growth of the resource is of compensatory nature. Thus the choice of optimal singular solution and the nature of associated approach paths make the problem pertinent and interesting.

1   Introduction

Allee effect generally refers to a reduction in individual fitness at low population size or density [1]. This effect plays a very important role in the dynamics of populations. There are several mechanisms which generate Allee effect. Berec et al. [2] presents a classification of various Allee effects characterised by the nature of involved mechanisms. The history of Allee effect and its future importance has been discussed in [3]. There are quite a few real world examples exhibiting presence of Allee effects [1,4,5]. As a consequence, analysis of systems involving Allee effect has gained lot of importance in real world problems associated with various fields such as conservation biology [6,7], sustainable harvesting [8], pest control, biological control [9], population management [2], biological invasions [10], meta population dynamics [11,12], interacting species [13,14]. The review article [15] presents a classification of single species models that are subjected to various Allee effects. Most of the available literature deals with the influence of Allee effect on the system dynamics and very little is the associated bio-economics. In the language of bio-economics, if the growth of a resource is represented by a critical depensation curve then it is said to exhibit strong Allee effect [16]. Bio-economics of a resource with logistic growth or compensatory growth has been systematically developed in the literature [16,18]. Bio-economics of a resource in a seasonally varying environment has been recently presented in [17] wherein the growth of the resource is assumed to follow either logistic law or Gompertz law with periodic coefficients. Such comprehensive results do not appear to be known for resources subjected to Allee effect, although some results are established for certain specific dynamic models [8,19,20]. The present study deals with certain bio-economic aspects of a renewable resource subjected to strong Allee effect. The aim is to derive optimal exploitation strategies for the considered resource from sole owner perspective.

2   Single Species Models

Before we start discussing some fundamental aspects in Mathematical modelling let us recall some definitions and results from the theory of ordinary differential equations.

A general first order initial value problem is given by

$$y^0 = f(t,y),\ y(0) = y_0. \qquad (1.1)$$

Note that the above differential equation (1.1) is a non autonomous equation due to involvement of the (independent) variable, t, in the right hand side. In this entire course we are going to consider either first order differential equation or first order differential system of autonomous nature only. Therefore there is going to be no explicit involvement of the variable t in RHS. Also unless and otherwise stated we assume that all the ode's satisfy the picard's theorem. Hence every IVP admits a unique solution.

Now let us consider the following first autonomous differential equation (system)

$$\frac{dN}{dt} = f(N). \quad (1.2)$$

We have the following definitions:

Definition 1: All the solutions of the equation $f(N) = 0$ are called equilibrium solutions of the equation (1.2). These solutions are also some times called as equilibrium points, critical points, stationery points, rest points or fixed points.

Example: $x^* = 2$ and $x^* = 1$ are two critical points of the DE $x^0 = x^2 - 3x + 2$. It is easy to verify that $x^* = 2$ and $x^* = 1$ satisfy the DE $x^0 = x^2 - 3x + 2$.

Therefore if N^* is an equilibrium solution of the DE $N^0 = f(N)$ then $N(t) = N^*$ is the unique (constant) solution of the IVP $N^0 = f(N), N(t_0) = N^*$.

Thus, note that the equilibrium solutions are special constant solutions of the associated differential equation.

Definition 2: An equilibrium solution N^* of (1) is said to be Lyapunov stable, if for any given $^2 > 0$ there exists a $\delta > 0$ (depending on ^{2}) such that, for all initial conditions $N(t_0) = N_0$ satisfying $|N_0 - N^*| < \delta$, we have $|N(t) - N^*| < \ ^2$ for all $t > t_0$. Alternatively, we say that an equilibrium solution is said to be stable if solutions starting close to equilibrium solution (in a δ neighborhood) remain in the its ^{2} neighborhood for all future times.

Definition 3: An equilibrium solution N^* of (1.2) is

said to be asymptotically stable if • it is stable •

if there exists a $\rho > 0$ such that for all N_0 such

that $|N_0 - N^*| < \rho \Rightarrow \lim |N(t) - N^*| = 0$.

Alternatively, an equilibrium solution is said to be asymptotically stable if it is stable and in addition all solutions initiating in a ρ neighborhood of the equilibrium solution approach the equilibrium solution eventually.

Definition 4: A solution of (1.2) is said to be unstable if it is not stable.

The following theorem characterizes the asymptotic stability and instability of an equilibrium solution of (1.2).

Theorem: Suppose that N^* is an equilibrium point of the differential equation $N^0 = f(N)$, where $f(N)$ is assumed to be a continuously differentiable function with $f^0(N^*) \ 6= \ 0$. Then the equilibrium point N^* is asymptotically stable if $f^0(N^*) < 0$, and unstable if $f^0(N^*) > 0$.

Now, let us attempt to model dynamics of a single species. Let us assume that $N(t)$ represents the total number of individuals in a population or density of a population in an environment. We know that the term $\frac{dN}{dt}$ represents rate of change in the entire population while the term $\frac{1}{N}\frac{dN}{dt}$ represents per capita rate of change in the population (change per an individual in some loose sense). To start with, we assume that the change in the population is caused due to two processes only and they are births and deaths. If per capita birth and death rates are given by b and d respectively, then we can represent per capita rate of change by the difference between the above birth and death rates i.e.,

$$\frac{1}{N}\frac{dN}{dt} = b - d. \quad (1.14)$$

By representing $b - d$ as r the above relation can be alternatively presented as

$$\frac{dN}{dt} = rN. \quad (1.15)$$

In general, r is termed as *intrinsic growth rate*. If the initial population at time $t = t_0$ is N_0 then we have the following initial value problem (mathematical model) representing the dynamics of the population.

$$\frac{dN}{dt} = rN, N(t_0) = N_0. \quad (1.16)$$

The above model, which is called as *exponential model or Malthusian model* can be read from several perspectives.

- The per capita growth rate, $\frac{1}{N}\frac{dN}{dt}$, is always constant. (Contribution due to an average individual is always a constant, given by r and this contribution does not depend on the density of the population. See equation (1.14))

- The growth rate of the population, $\frac{dN}{dt}$, is always increasing (decreasing) if $r > 0 (r < 0)$. See equation (1.15)

The population grows (decays) exponentially from the initial value N_0. The population will remain constant only when the births and deaths balance each other i.e., $b = d$ or $r = 0$. See the IVP (10) whose solution is given by $N(t) = N_0 e^{rt}$.

From the above observations, we understand that, the model describes that the contribution due to an individual population is independent of the population present and if per capita birth rate b exceeds corresponding death rate d the population blows up to ∞ exponentially else it decays to zero in the same fashion. Assuming that the births exceed deaths i.e., $b-d = r > 0$, it is somewhat unrealistic to arrive at the conclusion that the population would blow to *infty*. This is due to our assumption that the per capita rate of change is independent of the total population. It is more realistic to assume that the per capita growth rate is a function of the total population and it decreases with the total population (understandably, as the population increases they have to share the limited food resources available. This naturally limits their growth.) Thus we modify our model by assuming that the per capita growth rate is a linearly decreasing function of the total population, given by,

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right). \tag{1.17}$$

Here, K is called as carrying capacity. This term represents the total number of population the environment can support. Observe that the per capita growth rate continuously reduces from r as the population N increases from zero and it becomes zero when the population reaches K. This seems reasonable as resources are always limited and the population are controlled by these resources.

Taking into consideration the initial population, we have the following modified model to represent growth in a single species, called as *logistic model*.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), N(t_0) = N_0. \tag{1.18}$$

Now we shall try to analyze the *logistic model* in the light of the theorem done earlier and try to understand the qualitative behaviour of the solutions of (1.18). Here, we have $f(N)$ to be

$rN(1 - \frac{N}{K})$. Equating this term to zero we obtain two equilibrium points, given by $N_1 = 0$ and $N_2 = K$. Since $f'(N) = r(1 - \frac{2N}{K})$ we have $f'(N_1) = r > 0$ and $f'(N_2) = -r < 0$ indicating that the equilibrium solution $N = 0$, which is also called as trivial solution, is unstable while the other equilibrium solution $N = K$ is asymptotically stable. Thus, solutions initiating in a neighborhood of K approaches K as $t \to \infty$ and no solution starting in a small neighborhood of 0 remains close to zero in the future. This behaviour can be well understood from the following analysis.

We have $\frac{dN}{dt} = rN(1 - \frac{N}{K}) = rN - r\frac{N^2}{K}$. Let us assume that, N_0 is very close to 0. Thus, N^2 will be much smaller and hence the term $\frac{N^2}{K}$ can be ignored in the above model. Therefore, we have

$$\frac{dN}{dt} \approx rN. \tag{1.19}$$

3   HARVEST MODELS Bifurcations and Break Points

A note on bifurcations: Bifurcation theory basically describes the way the topological features of a dynamical system vary as one or more parameters are varied. We illustrate these variations through a diagram called bifurcation diagram. Diagrams showing branching processes that illustrate how the location and stability of solutions depend on a parameter are called bifurcation diagrams. Below we take four examples to understand some basic features of bifurcations. Let us take x to be the state variable and μ to be a parameter, which we call as bifurcation parameter. We intend to study the variations in the nature of equilibrium solutions of various differential equations as the parameter μ is varied. Note that a bifurcation value of a parameter μ is a value at which the qualitative nature of the flow or the equilibrium solution changes.

4   Constant Rate Harvesting

Consider the population that is growing logistically and at a constant rate of harvesting h.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - h \tag{3.1}$$

Now, let us analyze the dynamics of this system. The equation (1) admits two equilibria points characterized by

$$f(N) = rN\left(1 - \frac{N}{K}\right) - h = 0$$

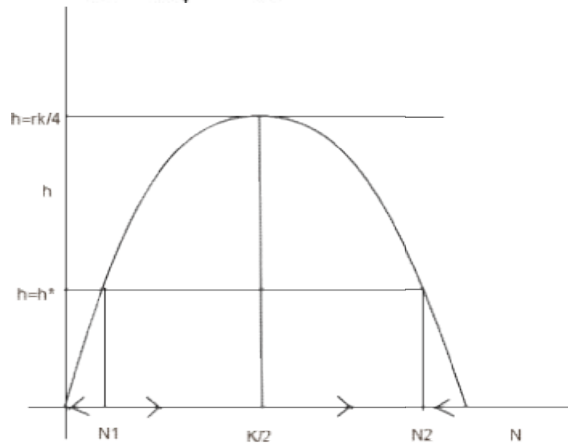$$\Rightarrow N = \frac{K}{2} \mp \frac{K}{2}\sqrt{1 - \frac{4h}{rK}}.$$

(3.2)

Let us name them as N_1 and N_2, where

$$N_1 = \frac{K}{2} - \frac{K}{2}\sqrt{1 - \frac{4h}{rK}}$$

and

(3.3)

$$N_2 = \frac{K}{2} + \frac{K}{2}\sqrt{1 - \frac{4h}{rK}}.$$

In order to find the nature of equilibrium points N_1 and N_2, we evaluate $f^0(N_1)$ and $f^0(N_2)$. Suppose

$$f(N) \quad = \quad l(N - N_1(h))(N_2(h) - N) \tag{3.4}$$

$$\Rightarrow f^0(N) \quad = \quad l(N_2(h) - N) - l(N - N_1(h)) \tag{3.5}$$

Therefore,

$$f'(N_1) = l(N_2(h) - N_1(h)) = r\sqrt{1 - \frac{4h}{rK}} > 0 \tag{3.6}$$

and

$$f'(N_2) = l(N_2(h) - N_1(h)) = -r\sqrt{1 - \frac{4h}{rK}} < 0 \tag{3.7}$$

The equilibrium point N_1 is unstable whereas the equilibrium point N_2 is asymptotically stable.

If suppose the harvest $h = 0$, then the equation (3.1) becomes logistic equation and admits two equilibria solutions $N_1 = 0$ and $N_2 = K$.
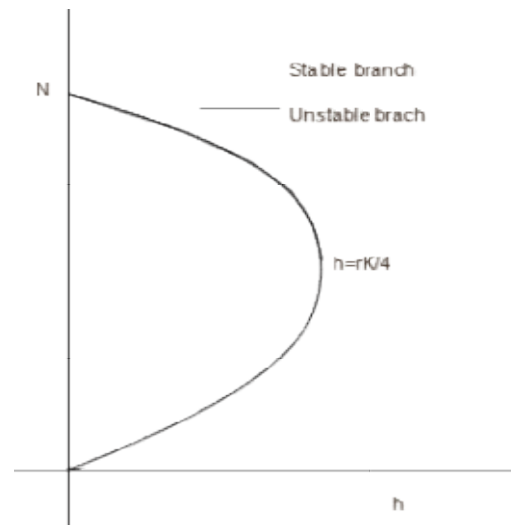
The two equilibria will coincide only when $1 - \frac{4h}{r} = 0$.

$$\Rightarrow h = \frac{rK}{4} \text{ and } N = \frac{K}{2}.$$

That is, when $N = \frac{K}{2}$, the two equilibria coincide and the point is called critical point.

When the harvest $h > \frac{rK}{4}$, then there are no equilibrium point and the entire system collapse.

Thus there occurs a saddle node bifurcation (as shown in fig 2)

The choice of harvesting h, plays a role in the sustaining of stock.

If suppose the harvest is made at $h = h*$ (as shown in fig 1), we obtain two equilibria N_1 and N_2.

If suppose the stock is to the left of N_1, then slowly, the density of population approaches zero as time progresses. But if the stock is to the right of N_1 and left of N_2, the density of population grows with time approaches N_2.

5   Harvest models and optimal control theory

In this chapter we study the interaction between humans and other renewable resources such as a fishery. Let us consider an open access fishery, by which we mean it is a fishery which is not controlled by any regulatory agency like the government. Any one (fisher men) can come utilize the resource and leave the fishery. Under such circumstances we wish to model the dynamics of the fishery and also the effort used for fishing. We assume that there is a cost for effort like boat hire, man power etc. If $N(t), E(t)$ represent the stock and effort at any time t and assuming that the stock is governed by the logistic law in the absence of harvesting and that the catch is proportional to the effort and the stock, we have the governing dynamic equation for the stock to be

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qEN.$$

(12.1)

Now let us try to derive an equation for the dynamics of the effort. If p stands for the price per unit catch and c is the cost per unit effort then at any time the profit or economic rent to the harvester is given by $pqE(t)N(t) - cE(t)$ and the fishery attracts other people (effort increases) as long as the economic rent is positive and the effort gets reduced once the cost exceeds the revenue (since the activity would no longer be lucrative). Therefore the change in the effort is proportional to the economic rent $pqE(t)N(t) - cE(t)$. Hence we have the dynamic equation for the harvest to be

$$\frac{dE}{dt} = k(pqEN - cE).$$

(12.2)

Observe that the above coupled equation is similar to that of prey-predator equation with stock as the prey and effort as predator although the effort equation has a different economic interpretation. In the light of the analysis done for the prey-predator coupled systems in the previous chapters, the dynamics of the fishery in presence of harvesting can be easily understood. Suppose the revenue (pqK), obtained from the catch by using a unit effort when the resource (fishery) is at its carrying capacity K, is less than the cost per unit effort c, then obviously the fishery is not economically viable. Hence no one goes for fishing and existing activity will be withdrawn. As a result the resource will reach its carrying capacity and effort goes to zero. We can observe this dynamics from the system too.

Now let us consider a sole owner fishery. Here the fishery is owned and regulated by a single individual. Here the objective of the owner is to maximize the (long time) profit during a time period $[0, T]$. In other words the owner wishes to find the harvesting strategy he/she has to implement during $[0, T]$ which will maximizes the profit from the harvesting activity. Thus the owner's problem is to derive an optimal harvesting strategy $E(t), t \in [0, T]$ satisfying the following:

(12.3a)

$$\max_{0 \le E(t) \le E_{max}} \int_0^T e^{-\delta t}[pqE(t)N(t) - cE(t)]dt$$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qEN, N(0) = N_0$$

subject to

(12.3b)

Expressing the above problem mathematically, the owner wishes to obtain a harvesting strategy $E(t)$ so as to maximize the discounted net economic rent over the period $[0, T]$ of his ownership. Here δ stands for discounting factor. This is a problem to be solved using *optimal control theory*. There are three problems we come across in optimal control theory which can be used to solve problems of the type mentioned above.

Now let us take an example problem and study its solvability by applying the maximum principles. Consider the following problem:

$$\max_{0 \le E \le E_{max}} \int_0^T pqE(t)N(t)dt$$

(12.11a)

subject to

$$\frac{dN}{dt} = f(N) - qEN, N(0) = N_0$$

(12.11b)

Observe that the above problem represents sole owner fishery with no costs and discounts. $N(t)$ is the stock level and $E(t)$ is effort. The Hamiltonian

associated with the optimal control problem is given by :

$$H(E,N,\lambda) = pqEN + \lambda[f(N) - qEN]$$
(12.12a)

alternatively

$$H = q(p - \lambda)EN + \lambda f(N) \quad (12.12b)$$

The associated canonical equations are:

$$\frac{dN}{dt} = \frac{\partial H}{\partial \lambda} = f(N) - qEN$$
(12.13a)

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial N} = -q(p - \lambda)E - \lambda f'(N)$$
(12.13b)

Observe that the Hamiltonian H is linear in the control variable $E(t)$. Thus the maximizer $E(t)$ of the hamiltonian satisfies

$E(t) = $ Emax, $\lambda(t) < p$; (12.14) 0, $\lambda(t) > p$.

There is another possibility where $\lambda(t) = p$ on an interval, I. In this case, the hamiltonian H becomes independent of E and this also implies that $\lambda^0(t) = 0$ on I. Comparing with the canonical equations this requires that $f^0(N)$ be zero on I. Thus, we require a N which not only maximizes the Hamiltonian but also satisfies the condition $f^0(N) = 0$ on I. Recall that for the dynamic equation

$$\frac{dN}{dt} = f(N) - qEN, \quad (12.15)$$

with $f(N) = rN\left(1 - \frac{N}{K}\right)$, we have the optimal effort E^*, MSY and the corresponding equilibrium N^* to be

$\frac{r}{2q}, \frac{rK}{4}, K\left(1 - \frac{qE^*}{r}\right)\left(= \frac{K}{2}\right)$ respectively.

Moreover, observe that $f^0(N*) = $

$r\left(1 - \frac{2N*}{K}\right) = 0$. Thus we have $N_{MSY}*$ maximizing the hamiltonian and satisfying $f^0(N) = 0$ on I. This implies that, on I, the corresponding effort level that is required to maintain the stock at $NMSY*$ is
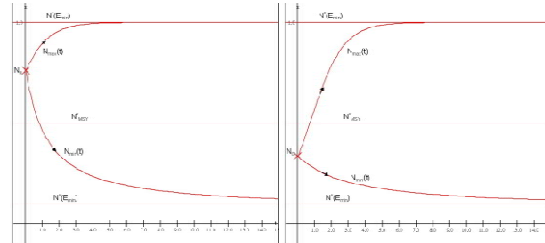
$$E_{MSY}^* = \frac{f(N_{MSY}^*)}{qN_{MSY}^*} . \quad (12.16)$$

Now we have the following form for the maximizer:

□

□ E_{max}, $\lambda(t) < p$;

$E(t) = $ □ $E_{MSY}*$ $\lambda(t) = p$; (12.17)

0, $\lambda(t) > p$.

From the above expression for the maximizer we have the complete information about the optimizer if we have the knowledge of the costate variable $\lambda(t)$, for which we need to solve the canonical equations. Now, let us consider the transversality condition **c** in the maximum

principle. From the considered problem, we have $G = 0, t_0 = 0, t_1 = T, x_0 = N_0$ and the value of the state N at $t = T$ is unconstrained. Thus, we have $dG = dt_0 = dt_1 = dx_0 = 0$ and $dx_1 = dN(T)$ arbitrary. Substituting these in the transversality condition, we obtain $\lambda(T) = 0$. Thus we have the following boundary value problem.

$$\frac{dN}{dt} = f(N) - qEN$$
(12.18a)

$$\frac{d\lambda}{dt} = -q(p - \lambda)E - \lambda f'(N)$$
(12.18b)

with $t \in [0,T], N(0) = N_0, \lambda(T) = 0$. Here, the initial point of the state and the final value of the costate variable are fixed. The end point of the state variable is free and $\lambda(0)$ has to be chosen conveniently such that the unique solution of the BVP satisfies the given boundary conditions. The zero value of λ at T implies that $E(T) = E_{max}$ as $\lambda(T) < p$. Now we need to fix the initial value of the costate variable, which fixes the optimal effort at the beginning of the plan period.

Therefore, if $N(0) > N_{MSY}*(N(0) < N_{MSY}*)$, then it is possible to drive the state from N_0 to $N_{MSY}*$ in a finite time $\hat{t}$ that is, $N(\hat{t}) = N_{MSY}*$ by applying maximum (minimum) possible effort. Looking at the equations $\frac{dN}{dt} = f(N) - qE_{max}N, \frac{d\lambda}{dt} = -q(p - \lambda)E_{max} - \lambda f'(N)$ with $\lambda(T) = 0$ there exists a time $t = l < T$ such that

a unique solution of the system exists satisfying $N(T - l) = N^*_{MSY}, \lambda(T - l) = p, \lambda(T) = 0$. That is, l is the time taken for the adjoint variable to reduce its value from p to 0 if the initial value of the state is at N_{MSY}^*.
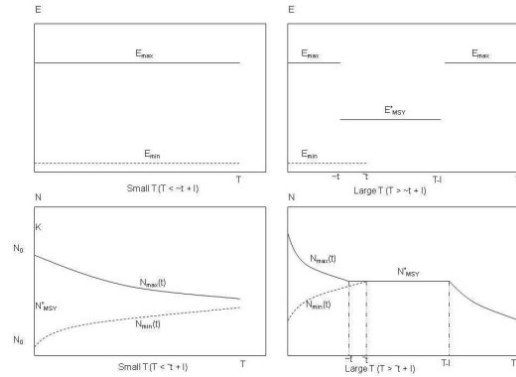
Now let us suppose that $0 < \lambda(0) < p$. Hence $E(0) = E_{max}$. Also we have $f^0(N_0) < 0$. Note that $-q(p - \lambda)E - \lambda f^0(N) = 0$ implies that $N = N_{MSY}^*$ and $\lambda = p$. In view of the canonical equation for λ,

$$\frac{d\lambda}{dt} = -q(p - \lambda)E - \lambda f'(N) = -qpE + \lambda(qE -$$
)), observe that if λ is in the positive neighborhood of 0 then the RHS of $\frac{d\lambda}{dt}$ will be negative as the term $-qpE$ dominates. On the other hand if λ is in the vicinity of p then the RHS of $\frac{d\lambda}{dt}$ will be positive. We need to choose the value of $\lambda(0)$ in such a way that $\lambda(T) = 0$. There are two possible ways to ensure that $\lambda(T) = 0$. If $\tilde{t} + l > T$ then choose $\lambda(0)$ so close to 0 so that $\frac{d\lambda}{dt} < 0$ and $\lambda(t)$ decreases to zero as t approaches T. Hence the corresponding optimal harvesting strategy is to apply maximum effort E_{max} during the entire plan period. On the other hand if $\tilde{t} + l < T$, choose $\lambda(0)$ so close to p so that $\lambda(\tilde{t}) = P$, (hence the corresponding harvesting effort is $E(t) = E_{max}$), and from $t = \tilde{t}$ switch the harvesting effort to E_{MSY}^* which maintains the stock at the maximum sustainable yield level and parallelly $\frac{d\lambda}{dt}$ at zero level and switch the effort $E(t)$ back to E_{max} at an appropriate time, that is at $t = T - l$, so that $\frac{d\lambda}{dt}$ turns negative again as a result value of λ reduces from p to 0 by the time t reaches T. Thus, if $N_0 > N^*_{MSY}$ then the optimal effort policy is as follows:

If $T > \tilde{t} + l$ then

$$E(t) = \begin{cases} E_{max} & t \in [0, \tilde{t}) \\ E_{MSY^*} & t \in [t, \tilde{T} - l] \\ E_{max} & t \in [T - l, T] \end{cases}$$

else

$$E(t) = E_{max}, t \in [0, T].$$

Similarly, if $N_0 < N_{MSY}^*$ it can be shown that the corresponding optimal harvesting strategy is as follows:

If $T > \bar{t} + l$ then

$$E(t) = \begin{cases} E_{min} & t \in [0, \bar{t}) \\ E_{MSY}^* & t \in [\bar{t}, T - l) \\ E_{max} & t \in [T - l, T] \end{cases} \text{(12.20a)}$$

Else

$$E(t) = E_{max}, t \in [0, T].$$

This model illustrates the fundamental principle of renewable resources given by **Larger discount rates implies Less biological conservation**. Now let us consider the case where the intrinsic growth rate of the resource, δ is greater than the discount rate δ. Here $f^0(N) = \delta$ has no solution. This suggests that the optimal solution in this case is to drive the source to extinction. This is for the simple reason that the values of the stock grows slower than the rate at which corresponding money would grow if invested in a the bank.

6   Discussion and conclusions

The optimal harvest policy is a feedback policy and coincides with most rapid approach path whenever the initial state is greater than the Allee threshold value of the resource. Construction of suboptimal approach paths is dependent on the stability nature of the optimal singular solution and the initial state of the resource. Whenever the optimal singular solution happens to be asymptotically stable and if the initial state is contained in its region of attraction (D), then the optimal singular solution can be reached asymptotically under constant singular harvest policy. If the initial state does not belong to D then the state has to be driven into D with appropriate control before the constant singular effort can be implemented. If the initial state is below the Allee threshold value of the resource, then no admissible control can retain the state at a positive level. In this case driving the resource to extinction by

harvesting with maximum effort is optimal. This policy becomes optimal harvest policy even for the case where the considered control problem does not admit any admissible singular solutions.

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