BASIS OF EQUIVALENCE AND EQUIVALENCY OF CAUCHY-SCHWARZ AND BESSEL INEQUALITIES IN VECTOR SPACE & INNER PRODUCT SPACE

L.Vijaya Kumar

Faculty of Mathematics, Dr.B.R.Ambedkar University, Srikakulam, Andhra Pradesh

Abstract: In this paper it is proved that the Cauchy-Schwarz and Bessel Inequalities in Real number system of vector space and as well as in inner product space.

Kew words: Cauchy-Schwarz Inequalities, Bessel Inequalities, Liner independent, linear combination, linear space, vector space, orthonormal, inner product space

1. INTRODUCTION

In the historical and logical point of view in Mathematics, the equivalency of inequalities plays a prominent role. There are the lot of articles regarding this topic. If an equivalence relation is defined on a set then the set can be separated into classes by the convention that two elements belong to the same class if and only if they are equivalent and these classes are called equivalent classes, two equivalence classes are identical if they have an element in common. Each element belongs to one of the equivalence classes. If a set X has been divided into no overlapping subsets then an equivalence relation can be defined by letting '*x*' is related to '*y*' mean that *x* and *y* are in the same subjects. Equivalence is a proposition formed from two given propositions by connecting them by if and only if. Equivalence is true if both propositions are true of if both are false. The proposition for all triangles x, x is equilateral if and only if x is equilateral is true, since any particular triangle is either both equilateral and equiangular or is neither equilateral nor equiangular. Equivalence is also called as biconditional statement. Two statements are logically equivalent if they are equivalent because of their logical form rather than because of mathematical content.

2. CAUCHY- SCHWARZ, BESSEL INEQUALITIES IN VECTOR SPACE

Definition 2.1: Let $\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n$ are *n*-vectors, then the linear combination is $c_1\bar{a}_1 + c_2\bar{a}_2 + \dots\dots + c_n\bar{a}_n = \sum_{i=1}^{n} c_i\bar{a}_i$ where $c_1, c_2, \dots\dots, c_n$ are scalars.

Definition 2.2: Let $\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n$ are *n*-vectors, if $\sum_{i=1}^{n} c_i\bar{a}_i = 0$ where all scalars are zero then $(\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n)$ are all linear independent vectors.

Definition 2.3: Let $\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n$ are *n*-vectors, if $\sum_{i=1}^{n} c_i\bar{a}_i$=0, for some scalars are may be zero then $(\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n)$ are Linear dependent vectors.

Definition 2.4: Let $\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n$ are basis of the Linear space L and L is *n*-dimensional, if

1. $\bar{a}_1, \bar{a}_2 \dots\dots \bar{a}_n$ are linear independent.
2. Every $\bar{x} \in L$ can be written uniquely $\bar{x} = x_1\bar{a}_1 + x_2\bar{a}_2 + \dots\dots + x_n\bar{a}_n$

Definition 2.5: Let L be a Linear space, A scalar product $(\bar{a}, \bar{b})$ is a function $L \times L \to R$ with the following properties

1. $|\bar{a}.\bar{b}| \le |\bar{a}|.|\bar{b}|$
2. $|\bar{a} + \bar{b}| \le |\bar{a}| + |\bar{b}|$

   Here (1) is called Cauchy-Schwarz inequality and (2) is called Triangle inequality.

Definition 2.6: Let L be an *n*-dimensional linear space with scalar product (Euclidean space) then,

1. A basis $e_1, e_2 \dots\dots e_n$ are called orthonormal basis,

$$\text{If } (e_i, e_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Theorem 2.1: Let $(X, (\ ,\ ))$ be vector space on a Real numbers (*R*), then the Cauchy-Schwarz Inequality and Bessel inequality are Equivalent.

Proof: Given that $(X, (\ ,\ ))$ be vector space on a Real numbers (*R*),

The Cauchy-Schwarz Inequalities says that

For each $\bar{a}$ and $\bar{b}$ in X then

$$\left|\bar{a}.\bar{b}\right| \leq |\bar{a}|.\left|\bar{b}\right| \qquad \text{(i)}$$

Also, the equality holds in (i) if and only if $\bar{a}$ and $\bar{b}$ are linear dependent.

On the other side

For $\bar{a}$ in X and let $\{e_1, e_2 \dots \dots e_n\}$ be an orthogonal set in X

$$\text{i.e } \left(e_i, e_j\right) = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

Then the Bessel inequality says that

$$\sum_{i=1}^{n} |(\bar{a}, e_i)|^2 \leq |\bar{a}|^2 \qquad \text{(ii)}$$

And the equality holds if and only if $\bar{a}$ is a linear combination of $e_i's$ and after that on the other case one computes the coefficients at once,

$$\bar{a} = \sum_{i=1}^{n} (\bar{a}, e_i) e_i$$

Now that the Bessel inequalities is more powerful then the Cauchy-Schwarz inequality

If we know $\sum_{i=1}^{n} |(\bar{a}, e_i)|^2 \leq |\bar{a}|^2$

Then for any non zero vector $\bar{a}$ and $\bar{b}$

We have $\left|\left(\bar{a}, \frac{\bar{b}}{|\bar{b}|}\right)\right|^2 \leq |\bar{a}|^2 \qquad \text{(iii)}$

This yield $\left|(\bar{a}.\bar{b})\right| \leq |\bar{a}|.\left|\bar{b}\right|$

Moreover, the equality holds in (i) if and only if the equality hold in (iii)

Or $\qquad \bar{a} = \left(\bar{a}, \frac{\bar{b}}{|\bar{b}|}\right) \frac{\bar{b}}{|\bar{b}|}$

Which is equivalent to linear dependent of $\bar{a}$ and $\bar{b}$

Now it remains to be proved that equation (i) yields (ii)

Now assume equation (i), then for each $\bar{a}$ and for each orthogonal set $\{e_1, e_2 \dots \dots e_n\}$ in X then we have

$$\sum_{i=1}^{n} |(\bar{a}, e_i)|^2 = \sum_{i=1}^{n} \left|(\bar{a}, e_i) - \sum_{j \neq i} (\bar{a}, e_j)\, e_j e_i\right|^2$$

$$= \sum_{i=1}^{n} \left|\left(\bar{a} - \sum_{j \neq i} \left(\bar{a}, e_j\right) e_j, e_i\right)\right|^2$$

$$= \sum_{i=1}^{n} \left|\left(\bar{a} - \sum_{j \neq i} \left(\bar{a}, e_j\right) e_j\right)\right|^2$$

$$= \sum_{i=1}^{n} \left(\bar{a} - \sum_{j \neq i} \left(\bar{a}, e_j\right) e_j, \bar{a} - \sum_{k \neq i} (\bar{a}, e_k)\, e_k\right)$$

$$= \sum_{i=1}^{n} (|\bar{a}|^2 - \sum_{k \neq i} |(\bar{a}, e_k)|^2 - \sum_{j \neq i} \left|(\bar{a}, e_j)\right|^2 + \sum_{k,j \neq i} \left(\bar{a}, e_j\right) \left(\overline{\bar{a}, e_k}\right) (e_j, e_k)$$

$$= n|\bar{a}|^2 - \sum_{i=1} \sum_{j \neq i} \left|\left(\bar{a}, e_j\right)\right|^2$$

$$= n|\bar{a}|^2 - (n-1) \sum_{i=1}^{n} \left|\left(\bar{a}, e_j\right)\right|^2 \qquad \text{(iv)}$$

Which is yields (ii) clearly by equation (iv),and the equality holds in (ii)

If and only if for each $i = 1,2, \dots \dots, n$, equality hold in

$$\left|\left(\bar{a} - \sum_{j \neq i} \left(\bar{a}, e_j\right) e_j, e_i\right)\right| \leq \left|\left(\bar{a} - \sum_{j \neq i} \left(\bar{a}, e_j\right) e_j\right)\right|$$

By this is equivalent saying that for each $i = 1,2, \dots \dots, n$, that the vectors e_i and $\bar{a} - \sum_{j \neq i} \left(\bar{a}, e_j\right) e_j$ are linear independent that is, that the equation

$$\bar{a} = \sum_{i=1}^{n} (\bar{a}, e_i)\, e_i$$

This shows that the Cauchy inequality and Bessel inequality are equivalent.

3. CAUCHY- SCHWARZ, BESSEL INEQUALITIES IN INNER PRODUCT SPACE

Definition 3.1: The set of all sequences $x = (x_1, x_2, \dots \dots)$ of complex numbers, where $\sum_{i=1}^{\infty} |x_i|^2$ is finite. The sum is $x + y$ is defined as $(x_1 + y_1, x_2 + y_2, \dots \dots.)$ then the product ax as $(ax_1, ax_2, \dots \dots.)$ and the inner product as $(x, y) =$

$\sum_{i=1}^{\infty} \bar{x}_i y_i$ where $x = (x_1, x_2, \ldots\ldots)$ and $y = (y_1, y_2, \ldots\ldots)$

Definition 3.2: For any real function F and an orthogonal normalized system of real function $f_1, f_2 \ldots\ldots$ on an interval (a, b) then the Bessels inequality is

$$\int_a^b [F(x)]^2 dx \geq \sum_{i=1}^{n} \left[\int_a^b F[x] f_i(x) dx\right]^2$$

Or for complex valued functions

$$\int_a^b [F(x)]^2 dx \geq \sum_{i=1}^{n} \left[\int_a^b F[x]\ \overline{f_i(x)} dx\right]^2$$

These are valid for all *n* if the function $F, f_1, f_2 \ldots\ldots$ are assumed to be Riemann integral.

Definition 3.3: For a vector space with an inner product space (*x*,*y*) and an orthogonal normalized set of vectors $\bar{a}_1, \bar{a}_2 \ldots\ldots \bar{a}_n$ then the Bessels inequality is

$$(u, u) = |u|^2 \geq \sum_{k=1}^{n} |(u, \bar{a}_k)|^2$$

Definition 3.4: The square of the integral of the product of two real functions over a given interval or region is equal to, or less then, the product of the integral of their squares over the same interval or region, provided these integral exists. For complex functions $f(z)$and $g(z)$

$$\bar{a}_n \left| \int_{z_1}^{z_2} |fg|^2 \leq [\int_{z_1}^{z_2} \bar{f} f |dz|][\int_{z_1}^{z_2} \bar{g} g |dz| \right|$$

When $\bar{f}$and $\bar{g}$are the complex conjugate of f and g, this inequality is easily deduced from Cauchy's inequality. It is also called the Cauchy-Schwarz inequality

For a vector space with an inner product (*x*,*y*) defined the inequality

$|\langle \bar{a}.\bar{b}\rangle| \leq \|\bar{a}\| \|\bar{b}\|$ is called Schwarz's inequality. In Hilbert space this inequality equal to the above inequality and to Cauchy inequality.

Note: Here Hilbert means complete inner product space.

Theorem 3.1: Let $(X, \langle, \rangle)$ be a Real or complex inner-product space, then the Cauchy-Schwarz Inequality and Bessel inequality are Equivalent.

Proof: Given that $(X, \langle, \rangle)$ be inner product space

By Cauchy-Schwarz inequality $|\langle \bar{a}.\bar{b}\rangle| \leq \|\bar{a}\| \|\bar{b}\|$

And by Bessel's inequality

For $\bar{a}$ in X and let $\{e_1, e_2 \ldots\ldots e_n\}$ be an orthogonal set in X

$$\text{i.e } (e_i, e_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Then the Bessel inequality says that

$$\sum_{i=1}^{n} |\langle \bar{a}, e_i\rangle|^2 \leq \|\bar{a}\|^2$$

Then the equality holds if and only if $\bar{a}$ is a linear combination of $e_i's$ and after that on the other case one computes the coefficients at once,

$$\bar{a} = \sum_{i=1}^{n} \langle \bar{a}, e_i\rangle e_i$$

And then $\sum_{i=1}^{n} |\langle \bar{a}, e_i\rangle|^2 = \sum_{i=1}^{n} \left|\langle \bar{a}, e_i\rangle - \sum_{j \neq i} \langle \bar{a}, e_j\rangle\, e_j e_i\right|^2$

Then we solve we get $\sum_{i=1}^{n} |\langle \bar{a}, e_i\rangle|^2 = n\|\bar{a}\|^2 - (n-1) \sum_{i=1}^{n} \left|\langle \bar{a}, e_j\rangle\right|^2$

∴ In inner product space the two inequalities are equivalent.

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